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EDITED BY

EARLE RAYMOND HEDRICK
WALLIE ABRAHAM HURWITZ JOHN WESLEY YOUNG

WITH THE ASSISTANCE OF

ROBERT WILBUR BURGESS DERRICK NORMAN LEHMER
DUNHAM JACKSON TULLIO LEVI-CIVITA
EDWARD KASNER HENRY LEWIS RIETZ

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REPORT ON CONTINUOUS CURVES FROM THE VIEWPOINT OF ANALYSIS SITUS*

BY R. L. MOORE

1. *The Characterization of a Continuous Curve.* A *continuous curve* is the set of points represented by the pair of equations

$$x = f_1(t), \qquad y = f_2(t),$$

where $f_1(t)$ and $f_2(t)$ are continuous functions of t on the interval $I : (0 \leqq t \leqq 1)$. Thus a continuous curve is the image of the straight line interval I under a continuous † transformation which transforms each point X of the interval I into a single point $T(X)$ of the curve. In case this transformation does not throw any two distinct points of I into the same point of the curve $T(I)$, then $T(I)$ is a *simple continuous arc.* Thus a simple continuous arc is in one to one continuous correspondence ‡ with a straight line interval. If $T(0) = T(1)$, but no point of $T(I)$ except $T(0)$ is the transform, under T, of more than one point of I, then $T(I)$ is a *simple closed curve.* It easily follows that a simple closed curve is in one to one continuous correspondence with a circle.

It is clear from the above definition that every continuous

* Presented before the Southwestern Section of the Society, in somewhat different form, at the Symposium held in Lawrence, Kansas, Dec. 2, 1922.

† The point P is said to be a *limit point* of the point set M if, for every positive number ϵ, there are points of M, distinct from P, at a distance from P less than ϵ. A transformation T which throws a point set M into a point set $T(M)$ is said to be *continuous* if, in case the point P of M is a limit point of a point set N which is a subset of M, then $T(P)$ is a limit point of $T(N)$.

‡ Two point sets M and N are said to be in *one to one correspondence* if there exists a correspondence in which (a) to each point P of M there corresponds just one point P' of N, (b) no two distinct points of M correspond to the same point of N, (c) if the point P' of N corresponds to the point P of M, then the point P of M corresponds to the point P' of N and conversely. Such a correspondence is sometimes called a one to one *reciprocal* correspondence. For so-called one to one correspondences which are not reciprocal I prefer to use the term transformation which seems to me to be more suggestive of an operation which is thought of as taking place in one direction.

curve is a bounded continuum.* That it is not, however, very easy to determine directly from this definition whether or not even some of the simplest continua are continuous curves will, I think, be admitted if one recalls the interest aroused by Peano's discovery of the fact that a square plus its interior is such a curve and the proofs of this fact that were given by Peano, Hilbert and E. H. Moore.†

In 1908, Schoenflies established a theorem which embodies a point set theoretic characterization (definition) of a continuous curve. This is, apart from minor matters of phraseology, etc., as follows: In order that a bounded continuum M should

* A set of points is said to be *closed* if it contains all its limit points. A point set is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one. (Cf. N. J. Lennes, AMERICAN JOURNAL, vol. 33 (1911), pp. 287–326.) A set of points is said to be *bounded* if it lies wholly in the interior of some circle. A *continuum* is a closed and connected point set.

† Peano, *Sur une courbe, qui remplit toute une aire plane*, MATHEMATISCHE ANNALEN, vol. 36 (1890); Hilbert, *Ueber die stetige Abbildung einer Linie auf ein Flächenstück*, ibid., vol. 38 (1891), pp. 459–460; E. H. Moore, *On certain crinkly curves*, TRANSACTIONS OF THIS SOCIETY, vol. 1 (1900), pp. 72–90. That a square plus its interior is not a simple continuous arc had been proved by Netto about ten years before. (E. Netto, *Beitrag zur Mannigfaltigkeitslehre*, JOURNAL FÜR MATHEMATIK, vol. 86 (1879), pp. 263–268.) On page 329 of the 1907 edition of his *The Theory of Functions of a Real Variable*, Hobson gives a proof of this theorem. Referring to the proofs of Netto and of Loria, he says: "In the proof given by these writers it is assumed that a closed curve corresponds to a linear sub-interval of (0, 1); this is not necessarily the case, for a non-dense closed set may correspond to the closed curve." This statement of Hobson's (which is repeated in the 1921 edition of his treatise) is manifestly incorrect. Neither a simple closed curve nor any other sort of connected point set could possibly be in one to one continuous correspondence with a non-dense closed set or any other set which is not connected. As far as I know, I have never seen anywhere in the literature the following extremely simple proof of Netto's theorem. Let X denote a point, other than an end-point, of the interval I. Suppose there exists a continuous one to one correspondence between I and the point set K constituted by a square plus its interior. Let X' denote the point of K which corresponds to X. The set $I - X$ is not connected. Hence $K - X'$ is not connected. But clearly $K - X'$ *is* connected. Thus the supposition that the correspondence in question exists leads to a contradiction. The same proof shows that there is no one to one continuous correspondence between the interval I and the set of points constituted by a simple closed curve.

be a continuous curve it is necessary and sufficient that (*a*) for every given positive number ϵ there are not more than a finite number of complementary domains * of M of diameter greater than ϵ, (*b*) the boundary of every complementary domain of M is accessible † at each of its points, from all sides, with respect to that domain.

That the same class of point sets would not be characterized by this definition if accessibility from all sides were replaced by mere accessibility may be seen with the help of the following example. Let M_1 denote the point set composed of the straight line intervals AB, BC, CD, where the coordinates of the points A, B, C, and D are $(0, 1)$, $(0, -2)$, $(1/\pi, -2)$ and $(1/\pi, 0)$ respectively. Let M_2 denote that portion of the curve $y = \sin(1/x)$ which lies between the lines $x = 0$ and $x = 1/\pi$. Let M_3 denote the point set $M_1 + M_2$. The point set M_3 separates the plane into two domains K and H, where H is bounded. Let M denote the continuum $H + M_3$. The only domain complementary to M is K. Furthermore M_3, the

* A connected point set K is said to be a domain if for each point P of K there exists a positive number δ_P such that K contains every point whose distance from P is less than δ_P. The boundary of a point set K is the point set composed of all points $[X]$ such that for each positive number ϵ there is at least one point of K and at least one point which does not belong to K, at a distance from X less than ϵ. A complementary domain of a closed point set M is a domain K which contains no point of M but whose boundary is a subset of M. The diameter of a bounded point set K is a number d such that (*a*) if X and Y are any two points of K then the distance from X to Y is not greater than d, (*b*) for every positive number e there exist points X and Y belonging to K such that the distance from X to Y differs from d by less than e.

† The boundary B of a domain K is said to be accessible at the point P with respect to K (and P is said to be accessible from K) if, for every point A in K, there exists a simple continuous arc AP which lies, except for the point P, wholly in the domain K. Suppose that XYZ is a simple continuous arc whose end-points X and Y lie on B, but every other point of which lies in the domain K. If B is connected then K is simply connected and is separated by the point set $XYZ - (X + Y)$ into two mutually exclusive domains, K_1 and K_2, such that $K = XYZ - (X + Y) + K_1 + K_2$. At least one of these domains (call it D_{XYZ}) has the point P on its boundary. If, for every arc XYZ satisfying the conditions stipulated, the point P is accessible from D_{XYZ}, P is said to be accessible from K from all sides and B is said to be accessible at P, from all sides, with respect to K.

boundary of K, is accessible at each of its points with respect to K. That it is not however accessible at every point from all sides with respect to K may be seen as follows. Let E denote the point $(1/\pi, 2)$ and let AED denote the simple continuous arc composed of the straight line intervals AE and ED. The origin O is not accessible from both of the domains into which K is divided by the point set $AED - (A + D)$, although O lies on the boundary of each of these domains. Thus O is not accessible from all sides with respect to K. Hence, by Schoenflies' theorem, M is not a continuous curve.

While Schoenflies' definition holds good for continuous curves in the plane, it does not hold for those in three dimensions. In fact I have recently shown * that, in order that, in space of three dimensions, a continuum M should be a continuous curve, it is not sufficient that it should have the two properties, (a) and (b), stipulated in Schoenflies' definition and it is not necessary that it should have *either* of them.

In 1914 Hans Hahn showed that in order that, in space of any number of dimensions, a bounded continuum should be a continuous curve, it is necessary and sufficient that it should have a property which he designates as that of *connectedness im kleinen.*† A continuum is said to be *connected im kleinen*

* *On the relation of a continuous curve to its complementary domains in space of three dimensions*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 33–38. On page 38 of this paper there is a theorem, involving conditions concerning the complementary domains of a bounded continuum M, in space of three dimensions, which are sufficient, but not necessary, in order that M should be a continuous curve.

† Hans Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), pp. 318–322. See also, however, S. Mazurkiewicz, *Sur les lignes de Jordan*, FUNDAMENTA MATHEMATICÆ, vol. 1 (1920), pp. 166–209. In this paper Mazurkiewicz introduces the same conception and establishes the same result, and refers, in this connection, to earlier papers of his own as follows: *O arytmetyzacji kontinuow*, C. R. Soc. Sc. VARSOVIE, vol. 6 (1913); *O arytmetyzacji kontinuow*, II, ibid., vol. 6 (1913); *O pewnej klasyfikacji punktow lezacych na kontinuach dowolnych*, ibid., vol. 9 (1916). I do not have access to these papers. The conception of *connectedness im kleinen*, as applied to a simple closed curve, was used by Pia Nalli, in the paper *Sopra una definizioni di dominio piano limitato da una curva continua, senza punti multipli*, RENDICONTI DI PALERMO, vol. 32 (1911), pp. 391–401.

at the point P if for every positive number ϵ there exists a positive number $\delta_{P\epsilon}$ such that if X and Y are any two points at a distance from P less than $\delta_{P\epsilon}$ then they lie together in a closed and connected subset of M every point of which is at a distance less than ϵ from the point P. This definition of a continuous curve has, for many purposes, a decided advantage over that of Jordan. For instance, it is easy to see that a square plus its interior (or, in three dimensions, a cube plus its interior) satisfies this definition and is therefore a continuous curve. Consider, on the other hand, the point sets M_1, M_2, M_3, described as follows:

Let M_1 denote the point set obtained by adding to the interval from $(0, -1)$ to $(0, 1)$ that part of the curve $y = \sin(1/x)$ which lies to the right of the y-axis and is bounded on the right by the line $x = 1$. Let M_2 denote the point set consisting of the intervals OB, OB_1, OB_2, OB_3, $\cdots$ where O is the origin of coordinates, B is the point $(0, 1)$ and, for every n, B_n is the point $(1/n, 1)$. Let M_3 denote the point set obtained by adding to the set M_1 the set of all the intervals $A_{mn}B_{mn}$ where, for every pair of positive integers m and n, A_{mn} and B_{mn} denote the points $(0, m/n)$ and $(m/n, m/n)$ respectively. I venture to surmise that one who is acquainted with no other characterization of a continuous curve than that of Jordan would probably not be able to decide, without some difficulty, the question, which of the sets M_1, M_2, M_3 is a continuous curve and which are not.* It is easy, however, to see that neither M_1 nor M_2 is connected im kleinen at the point $(0, 1/2)$, while M_3 is connected im kleinen everywhere. The set M_3 is accordingly a continuous curve, while M_1 and M_2 are not.

Mazurkiewicz, Tietze, and I, working, as far as I know, entirely independently of each other, have shown † that, in

* For examples in space of three dimensions see my paper *On the relation of a continuous curve to its complementary domains in space of three dimensions*, loc. cit.

† S. Mazurkiewicz, loc. cit., H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 284–291; R. L. Moore, *A theorem concerning*

order that a bounded continuum should be a continuous curve it is necessary that it should be arc-wise * connected. That this condition is not sufficient is shown by the existence of the above-described point set M_2, which, though arc-wise connected, is not a continuous curve. However, I have recently established the following theorem which embodies a generalization of the above-mentioned result.

THEOREM A. *In order that a continuum M should be a continuous curve it is necessary that every maximal connected subset of an open subset of M should be arc-wise connected.*†

Less than a week before the date of the delivery of this address, Mr. R. L. Wilder showed ‡ that this condition is also *sufficient*. It therefore affords a complete characterization of a continuous curve. It may be of interest to see just how this condition fails to hold true for the set M_2. Let C denote the point $(0, 1/2)$. Though the set M_2 is arc-wise connected and $M_2 - C$ is an open connected subset of M_2, the points O and B cannot be joined by a simple arc which is a subset of $M_2 - C$.

Sierpinski has shown § that a continuous curve may be characterized as a bounded continuum M such that, for every positive number ϵ, M is the sum of a finite collection of closed and connected point sets each of which is of diameter less than ϵ.

Thus we have five characterizations of the notion *continuous curve*, that of Jordan (or perhaps it should be called a modifi-

continuous curves, this BULLETIN, vol. 23 (1917), pp. 233–236. This paper was presented to the American Mathematical Society, October 28, 1916.

* A point set M is said to be arc-wise connected if every two distinct points of M are the extremities of a simple continuous arc which lies wholly in M.

† R. L. Moore, *Concerning continuous curves in the plane*, MATHEMATISCHE ZEITSCHRIFT, vol. 15 (1922), pp. 254–260. The point set K is said to be an open subset of the set M if K is a subset of M and $M - K$ is either vacuous or closed. A *maximal* connected subset of a point set K is a connected subset of K which is not a proper subset of any other subset of K.

‡ This result forms a part of Mr. Wilder's dissertation for the degree of Ph.D. at the University of Texas.

§ W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, FUNDAMENTA MATHEMATICÆ, vol. 1 (1920), pp. 44–60.

cation of that of Jordan), that of Schoenflies, that of Hahn and Mazurkiewicz, that of Sierpinski, and lastly a characterization whose sufficiency and necessity have been established by Mr. Wilder and by the author, respectively. Of these characterizations, all but that of Schoenflies hold good in space of any number of dimensions.

2. *The Relation of a Continuous Curve to its Complementary Domains, in Space of two Dimensions.* Returning to Schoenflies' characterization, we may picture any continuous curve in the plane as the point set complementary to a finite or countably infinite set of simply connected domains such that (*a*) one of these domains is unbounded and contains, as a subset, the exterior of some circle which encloses the continuous curve, (*b*) for every positive number ϵ there are not more than a finite number of these domains of diameter greater than ϵ, (*c*) the boundary of each of these domains is, at each of its points, accessible from all sides with respect to that domain. Miss M. Torhorst has shown that the boundary of every complementary domain of a continuous curve is itself a continuous curve.* I have recently found that the outer boundary of every bounded complementary domain of a continuous curve is a simple closed curve and that if two points are separated from each other by a continuous curve M then they are also separated by some simple closed curve which is a subset of M.† From this result, with the aid of the fact ‡ that every simple closed curve separates the plane, it follows that in order that a non-dense continuous curve

* Cf. Marie Torhorst, *Ueber den Rand der einfach zusammenhängenden ebenen Gebiete*, MATHEMATISCHE ZEITSCHRIFT, vol. 9 (1921), pp. 44–65. In his thesis Mr. Wilder has shown that, indeed, every closed and connected subset of such a boundary is a continuous curve. This result extends, to the case of any such boundary, a proposition previously established by Mazurkiewicz for the case of a continuous curve which contains no simple closed curve. Cf. S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, FUNDAMENTA MATHEMATICÆ, vol. 2 (1921), pp. 119–130.

† *Concerning continuous curves in the plane*, loc. cit.

‡ Cf. O. Veblen, *Theory of plane curves in non-metrical analysis situs*, TRANSACTIONS OF THIS SOCIETY, vol. 6 (1905), pp. 107–112; and L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, MATHEMATISCHE ANNALEN, vol. 69 (1910), pp. 169–175.

should separate the plane it is necessary and sufficient that it should contain a simple closed curve.

I will say that a point set M has property S if, for every positive number ϵ, M is the sum of a finite number of connected subsets all of diameter less than ϵ. As applied to closed (and bounded) point sets this property is * equivalent to that of connectedness im kleinen. But, as applied to point sets which are not necessarily closed, it is † stronger than that of connectedness im kleinen and weaker than that of uniform ‡ connectedness im kleinen. Every domain is connected im kleinen. In order that a bounded and simply connected domain should have a continuous curve as its boundary it is § necessary and sufficient that it should have property S, while in order that it should have a simple closed curve as its boundary it is necessary and sufficient that it should have the stronger property of being uniformly connected im kleinen.‖

In his thesis Mr. Wilder makes a detailed study of the continuous curve which is the boundary of a domain. He has found that in order that the boundary B of a simply connected domain should be a continuous curve it is necessary and sufficient that every connected subset of B should be arc-wise connected.

3. *A Characterization of Continua which are not Continuous Curves.* I have found the following theorem very useful in the study of continuous curves.

THEOREM. *In order that a bounded continuum M should fail to be a continuous curve, it is necessary and sufficient that there should exist two concentric circles k_1 and k_2 and a countable*

* Cf. W. Sierpinski, loc. cit.

† Cf. my paper, *Concerning continuous curves in the plane*, loc. cit.

‡ A point set M is said to be uniformly connected im kleinen if, for every positive number ϵ, there exists a positive number δ_ϵ such that if X and Y are two points of M at a distance apart less than δ_ϵ then X and Y lie in some connected subset of M of diameter less than ϵ. Cf. Hans Hahn and S. Mazurkiewicz, loc. cit.

§ R. L. Moore, loc. cit.

‖ See my paper, *A characterization of Jordan regions by properties having no reference to their boundaries*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 4 (1918), pp. 364–370.

infinity of continua $\overline{M}$, M_1, M_2, M_3, $\cdots$, *such that* (1) *each of these continua is a subset of* M *and contains at least one point on* k_1 *and at least one point on* k_2, *and is a subset of the point set* H *which is composed of the two circles* k_1 *and* k_2 *together with all those points of the plane which lie between these circles,* (2) *no two of these continua have a point in common, and, indeed, no one of them is a proper subset of any connected point set which is common to* M *and* H, (3) *the set* $\overline{M}$ *is the sequential limiting set of the sequence of sets* M_1, M_2, M_3, $\cdots$,* (4) *there exists a connected subset of* M *which contains all the sets of the sequence* M_1, M_2, M_3, $\cdots$, *but which contains no point of the greatest connected point set which is common to* $\overline{M}$ *and* H.

For an indication of a proof of all but part (4) of this theorem see my papers, *Continuous sets that have no continuous sets of condensation,* and *A characterization of Jordan regions by properties that have no reference to their boundaries.*† The important part (4) has been supplied recently by Mr. Wilder. With its aid he established the above-mentioned theorem that in order that a bounded continuum M should be a continuous curve it is sufficient that every maximal connected subset of an open subset of M should be arc-wise connected. I have made use of parts (1), (2), and (3) on several occasions.

4. *A Continuous Curve in the Rôle of a Space.* It is of interest to note that if the set of points S which constitute a continuous curve is viewed as a space and certain terms are properly defined, then many of the theorems which hold true in ordinary space continue to hold true in S. For each point

* The point set M is said to be the limiting set of the sequence of point sets M_1, M_2, M_3, $\cdots$ provided that (*a*) each point of M is the sequential limit point of an infinite subsequence of some sequence of points P_1, P_2, P_3, $\cdots$ such that, for every n, P_n belongs to M_n, (*b*) if P_1, P_2, P_3, $\cdots$ is a sequence of points such that, for every n, P_n belongs to M_n, then M contains the sequential limit point of every subsequence of P_1, P_2, P_3, $\cdots$ that has a sequential limit point. If the further condition is satisfied that every infinite subsequence of the sequence M_1, M_2, M_3, $\cdots$ has the same limiting set M, then M is said to be the sequential limiting set of the sequence M_1, M_2, M_3, $\cdots$.

† This BULLETIN, vol. 25 (1919), pp. 174–176; and PROCEEDINGS OF THE NATIONAL ACADEMY, loc. cit.

P in S and each positive number ϵ, let $R_{P\epsilon}$ denote the set of all those points of S which lie, together with P, in a connected subset of S which lies wholly within a circle with center at P and radius equal to ϵ. For every P and ϵ the point set $R_{P\epsilon}$ will be called a *region with respect to* S. With the use of the fact that S is connected im kleinen, it may be seen that a point X in S is a limit point of a point set M which lies in S if and only if every region (with respect to S) which contains X contains also a point of M distinct from X. It follows that if S is identified with the S of my set Σ_1 of axioms * for plane analysis situs then the point X is a limit point of M in the ordinary sense if and only if it is a limit point of M in the sense defined in F. A. With the help of this fact it may be seen that Axioms 1, 2, and 4, and a number of the theorems of that paper hold true in the space S, the terms (other than region) that are used in the statement of those axioms and theorems being given meanings which are ordinarily attached to them when S is considered as a subset of ordinary space. Thus, corresponding to Theorem 15 of F. A., we have the previously mentioned theorem that every two points of a connected open subset † of a continuous curve can be joined by a simple continuous arc lying wholly in that subset.

In his thesis, in addition to other results in this connection, Mr. Wilder has established the following theorem.

THEOREM. *If S is a continuous curve in space of two dimensions, D is a domain with respect to S, B is the boundary of D with respect to S, and P is a point of D, then in order that P should be accessible, in S, from every point of D it is sufficient either (a) that B should be connected im kleinen, or (b) that B should contain no continuum of condensation ‡ which contains P.*

5. *The Non-Cut Points of a Continuous Curve.* A point P of

* *On the foundations of plane analysis situs*, TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164. This paper will be referred to hereafter as F. A.

† A connected open subset of S is called a domain in F. A., and such a subset of a continuous curve S may be called a domain with respect to S.

‡ A sub-continuum K of a point set M is said to be a *continuum of condensation* of M if every point of K is a limit point of $M - K$.

a connected point set M is said to be a *non-cut point* of M or a *cut point* of M according as $M - P$ is or is not connected. A cut point of M is said to disconnect M in the strong sense (or, merely, to disconnect M). A point set M is said to be connected in the strong sense if every two points of M lie together in some closed and connected subset of M. If $M - P$ is not connected in the strong sense then P is said to disconnect M in the weak sense. According to this definition, if a point P disconnects a point set M in the strong sense then it also disconnects it in the weak sense; but the converse is not always true. By Theorem A of § 1, if M is a continuous curve, then it is true, conversely, that if P disconnects M in the weak sense it also disconnects it in the strong sense.

In my paper *Concerning simple continuous curves*,* I proved that if a bounded continuum M does not contain more than two non-cut points then it is a simple continuous arc. But a simple continuous arc contains two non-cut points, namely its end-points. Thus we have the following theorem.

THEOREM C. *Every bounded continuum contains at least two non-cut points (i.e., two points neither of which disconnects it).*†

In his paper, *Un théorème sur les lignes de Jordan*,‡ Mazurkiewicz has established the following theorems.

THEOREM $\overline{\text{C}}$. *Every continuous curve contains at least two points neither of which disconnects it in the weak sense.*

THEOREM D. *If M is a continuous curve which contains a simple closed curve k then k does not contain an uncountable set of points each of which disconnects M in the weak sense.*

Theorem C is a logical consequence of Theorems A and C.

Let D′ denote the theorem obtained from Theorem D by the substitution of the word *strong* instead of the word *weak*. I have recently established the following theorems,§ of which

* Loc. cit. Concerning the close relationship between this result and a result announced sometime before by Janiszewski see page 340.

† J. R. Kline has shown that if a continuum has no non-cut points it is connected im kleinen. Cf. this BULLETIN, vol. 27 (1921), p. 399. That such a continuum is unbounded follows from Theorem C.

‡ Loc. cit.

§ For proofs of the first three of these theorems see my paper, *On the*

the first includes Theorem D′ as a special case but is much more general than Theorem D′.

THEOREM $\overline{\text{D}}$. *No closed and connected point set M (whether it be a continuous curve or not) contains a closed and connected subset K (whether K be a simple closed curve or not) which contains an uncountable set of points each of which disconnects * M but not K.*

THEOREM E. *In order that a bounded continuum M should be a continuous curve which contains no simple closed curve, it is necessary and sufficient that every sub-continuum of M should contain uncountably many points each of which disconnects M in the strong sense.*

THEOREM F. *In order that the continuous curve M should contain no simple closed curve it is necessary and sufficient that if K denotes the set of all those points of M that are not cut points of M, then no subset of K disconnects † M, even in the weak sense.*

THEOREM G.‡ *In order that a bounded continuum M should be a simple closed curve, it is necessary and sufficient that it should be disconnected by the omission of any two of its points.*

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cut points of continuous curves and of other closed and connected point sets, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 9 (1923), pp. 101–106.

* That this theorem does not remain true if the word *disconnects* is replaced by the phrase *disconnects in the weak sense* may be seen if one considers the example of a point set M which is composed of a circle K together with a suitably selected simple spiral which has every point of K as a limit point. In this example, no point of K disconnects K either in the strong or in the weak sense but every point of K disconnects M in the weak (but not in the strong) sense. This example also shows that Mazurkiewicz' Theorem D would not remain true if the phrase *continuous curve* were replaced by the phrase *bounded, closed, and connected point set.*

† If N is a subset of M and $M - N$ is not connected then N is said to disconnect M. If $M - N$ is not strongly connected then N is said to disconnect M in the weak sense.

‡ Cf. my paper *Concerning simple continuous curves*, loc. cit. In a paper presented to the Society but, as far as I know, not as yet published, J. R. Kline has shown that in this theorem the stipulation that the set M is bounded may be replaced by the stipulation that M has no cut point. He also characterizes a simple closed curve as a continuum which is not disconnected by the omission of any proper connected subset. Cf. this BULLETIN, vol. 27 (1921), p. 399, and vol. 28 (1922), p. 8.

6. *Concluding Remarks.* Beyond certain propositions which hold true for both two and three dimensions, comparatively little is known concerning continuous curves in three-dimensional space. Various results obtained by Schoenflies, Miss Torhorst and others concerning the relation of a continuous curve to its complementary domains in the plane do not hold true in three-space.* For instance Schoenflies has shown that if a bounded plane continuum divides its plane into just two domains S_1 and S_2 such that every point of M is accessible from both S_1 and S_2 then M is a simple closed curve. In three-dimensional space however there exists a bounded continuum M which divides space into just two domains S_1 and S_2 such that (*a*) every point of M is accessible from both S_1 and S_2 and (*b*) the domain S_1 is uniformly connected im kleinen, but M is neither a simple closed surface nor any sort of continuous curve. I have shown,† that such a point set M would be a continuous curve if *both* S_1 and S_2 were uniformly connected im kleinen. It seems to me conceivable that some results concerning continuous curves and their complementary domains may be modified that they will hold true for three-space if, in addition to other changes, one replaces in certain connections the diameter of a point set by what I have called the two-dimensional extent of a point set.

DEFINITION. A point set M is said to have a two-dimensional extent greater than e if there exists a right circular cylinder α with bases of diameter greater than e such that (*a*) neither base of α contains a point of M, (*b*) M contains at least one point in common with every simple continuous arc which has one end-point on one base of α, and its other end-point on the other base of α and lies, except for its end-points, wholly within α.

Every simple closed curve in a space S_2 of two dimensions can be thrown into any other simple closed curve in S_2 by a continuous one to one reciprocal transformation of S_2 into

* See my paper, *On the relation of a continuous curve to its complementary domains in space of three dimensions*, loc. cit.

† Loc. cit.

itself.* In other words in S_2 every two simple closed curves are *equivalent to each other*, in the strong sense, from the viewpoint of analysis situs. This proposition does not hold true for a space S_3 of three dimensions, as may be easily seen if one considers two simple closed curves in S_3 of which one is knotted and the other one is not. A knotted simple closed curve may be defined, from the viewpoint of analysis situs, as a simple closed curve (see definitions above) which is not a part of any simple closed † surface. It seems likely (*a*) that there exists no knotted simple continuous arc, that is to say, no simple continuous arc which does not lie on some simple closed surface, and (*b*) that any simple continuous arc in S_3 can be transformed into any other simple continuous arc in S_3 by a continuous one to one reciprocal transformation of S_3 into itself. But, as far as I know, no one has ever proved this. Indeed, as far as I know, there is nowhere in the literature any indication that anyone has *even considered* the question whether, if AB is an arc in a three-space S, and X is a point of S which does not belong to AB, then there exists an arc XB which has in common with AB only the point B. In other words, is an arc accessible at each of its ends from every point which does not lie on it?

A fruitful field for future investigation is afforded in that branch of analysis situs which has to do with various special types of continuous curves such as simple continuous arcs, simple closed curves and simple closed surfaces, in space of three dimensions. Little is known concerning the relation of such point sets to their complementary domains.

THE UNIVERSITY OF TEXAS

* Cf. A. Schoenflies, MATHEMATISCHE ANNALEN, vol. 62 (1906), p. 324. Also J. R. Kline, *A new proof of a theorem due to Schoenflies*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 6 (1920), pp. 529–531.

† By a simple closed surface is meant a point set which is in one to one continuous correspondence with the surface of a sphere. For a topological characterization of such a point set, see the abstract of an unpublished paper by J. R. Kline and the author, this BULLETIN, vol. 28 (1922), p. 380. As far as I know, there does not exist in the literature a proof that a simple closed surface is the image of a sphere under a one to one reciprocal continuous transformation of S into itself. See, however, an abstract by J. W. Alexander, this BULLETIN, vol. 28 (1922), p. 10.

A GENERALIZATION OF A PROPERTY OF AN ACNODAL CUBIC CURVE

BY HAROLD HILTON

1. *Introduction.* We refer to the following property.

*If in a plane cubic curve there is inscribed a real triangle ABC such that BC, CA, AB touch the curve at C, A, B, then the cubic can be projected by a real projection so as to have trigonal symmetry, i.e., it can be brought to self-coincidence by rotating it through $2\pi/3$ about a point. If, in particular, the cubic is unicursal (rational), it must be acnodal.**

The generalization suggested is that of any unicursal curve in which a triangle ABC is inscribed, so that A, B, C are each given by a single value of a parameter in terms of which the coordinates of any point of the curve are rationally expressed, while the intersections of BC, CA, AB with the curve lie respectively p at C and q at B, p at A and q at C, p at B and q at A. We shall investigate the properties of such curves.

Take the parameters of A, B, C as $0, \infty, 1$.† Then choosing suitable homogeneous coordinates, we have evidently

$$x : y : z = (t-1)^p : (-t)^p(t-1)^q : (-t)^q. \tag{1}$$

We shall find it convenient to use a quantity ϵ defined by

$$\epsilon \equiv p^2 - pq + q^2. \tag{2}$$

Elimination of t from (1) gives

$$x^{p/\epsilon}y^{q/\epsilon} + y^{p/\epsilon}z^{q/\epsilon} + z^{p/\epsilon}x^{q/\epsilon} = 0. \tag{3}$$

Hence the curves may be projected by a real projection so as to have trigonal symmetry, as in the case of the cubic. Points with parameters t, $1/(1-t)$, $(t-1)/t$ are those related by the symmetry. If $p = q$, p and q are factors of ϵ, and the curve is one of the "triangular-symmetric" curves discussed elsewhere.‡ We shall therefore suppose p and q unequal in

* For each non-singular or acnodal cubic, two such real triangles exist.

† See Hilton, *Plane Algebraic Curves*, Clarendon Press, p. 148. This book is referred to later as "H. P. A. C."

‡ MESSENGER OF MATHEMATICS, vol. 50, (1921), p. 171.

what follows, and there is no loss of generality in supposing $p > q$. The methods of page 138 of H. P. A. C. show that the curve has no cusps other than those involved in the singularities at A, B, C; and that the only inflexions other than those at A, B, C are the three real inflexions given by

$$pq(p-q)(t^3-3t+1) = \{(p^2-pq+q^2)(p+q-1) - 3pq^2\}(t^2-t). \tag{4}$$

The tangent at the point (1) is

$$t^p\{-pt+(p-q)\}x+(-1)^p(t-1)^{p-q}\{(p-q)t + q\}y+(-1)^q t^{p-q}(t-1)^p(qt-p)z = 0. \tag{5}$$

From each of the points A, B, C one real tangent can be drawn other than the sides of the triangle ABC. The parameters of the points of contact are $(p-q)/q$, $q/(q-p)$, p/q.

2. *Nodes.* Consider now the parameters t_1, t_2 of a node of the curve other than A, B, C. We have

$$\left(\frac{t_1-1}{t_2-1}\right)^p = \left(\frac{t_1}{t_2}\right)^p\left(\frac{t_1-1}{t_2-1}\right)^q = \left(\frac{t_1}{t_2}\right)^q. \tag{6}$$

Eliminating first $(t_1-1)/(t_2-1)$ and then t_1/t_2 from (6), we find that both these quantities are roots of the algebraic equation $x^\epsilon = 1$. We write therefore

$$t_2 = \omega t_1, \quad (t_2-1) = \Omega(t_1-1), \quad \text{where} \quad \omega^\epsilon = \Omega^\epsilon = 1. \tag{7}$$

Then (6) and (7) combined give

$$(1-\Omega)/(\omega-\Omega) \quad \text{and} \quad \varphi(1-\Omega)/(\omega-\Omega) \tag{8}$$

for the parameters of the node, subject to

$$\omega^\epsilon = \Omega^\epsilon = 1, \quad \Omega^p = \omega^q, \quad \Omega^q\omega^{p-q} = 1. \tag{9}$$

We now proceed to find the solutions of (9).

Let d be the highest common factor of p and q, and put

$$p = dp', \quad q = dq', \tag{10}$$

where p', q' are prime to one another. We have

$$\omega = e^{2k\pi i/\epsilon}, \quad \Omega = e^{2K\pi i/\epsilon}, \tag{11}$$

where k, K are any unequal integers included among 1, 2, 3, $\cdots$, $\epsilon - 1$, consistent with (9). Now (9) gives

$$\begin{cases} kq' - Kp' \equiv 0 \pmod{\epsilon/d}, \\ k(p'-q') + Kq' \equiv 0 \pmod{\epsilon/d}. \end{cases} \tag{12}$$

Elimination first of k and then of K from (12) shows that k and K are multiples of d. Put

$$k = dk', \qquad K = dK'. \tag{13}$$

Then the integers k', K' take all possible values 1, 2, 3, $\cdots$, $\epsilon/d - 1$, consistent with

$$k'q' - K'p' \equiv 0 \pmod{\epsilon/d^2} \tag{14}$$

excluding the cases in which k' and K' are equal and multiples of ϵ/d^2. Each value of k' which is not a multiple of ϵ/d^2 gives d values of K', and each value of k' which is a multiple of ϵ/d^2 gives $d - 2$ values of K'. Hence the number of possible values of ω and Ω is

$$d^2(\epsilon/d^2 - 1) + (d - 2)(d - 1) = \epsilon - 3d + 2.$$

Now the replacing of ω and Ω by $1/\omega$ and $1/\Omega$, respectively, merely interchanges t_1 and t_2. Hence the number of nodes other than A, B, C is $\frac{1}{2}(\epsilon - 3d + 2)$. If we replace ω, Ω by $1/\Omega$, ω/Ω we get a node obtained by the symmetry of the curve from that given by (7).

The coordinates of the nodes are got on replacing t in (1) by $e^{-k\pi i/\epsilon} \sin K\pi\epsilon \csc [(K - k)\pi/\epsilon]$, and $t - 1$ by $e^{-k\pi i/\epsilon} \sin k\pi\epsilon \times \csc [(K - k)\pi/\epsilon]$, where k and K are multiples of d, such that

$$kq/d^2 - Kp/d^2 \equiv 0 \pmod{\epsilon/d^2};$$

d being the highest common factor of p and q, and ϵ being $p^2 - pq + q^2$, as stated earlier. We readily see that the value of t is complex and that the ratios of the coordinates are real. Hence the point-singularities of the curve, excluding A, B, C, are $\frac{1}{2}(\epsilon - 3d + 2)$ real acnodes.

3. *The Rationalized Equation.* The equations § 1 (1) give the degree n as $p + q$, and (5) gives the class m as $2p - q + 1$ $(p > q)$. Moreover, the curve is unicursal. Therefore Plücker's equations give, with the notation of page 112 of H. P. A. C.,

$$\begin{gathered} n = p + q, \qquad m = 2p - q + 1, \\ \delta = \tfrac{1}{2}(p^2 + 2pq + q^2 - 3p - 9q + 8), \qquad \kappa = 3(q - 1), \\ T = \tfrac{1}{2}(4p^2 - 4pq + q^2 - 8p + 7q), \qquad \iota = 3(p - q). \end{gathered} \tag{15}$$

If we require the equation of the curve in rational form, we may proceed as follows. First suppose $d = 1$, so that p is

prime to q. Then from (3) the rationalized equation is

$$\prod(\Omega^p x^{p/\epsilon} y^{q/\epsilon} + y^{p/\epsilon} z^{q/\epsilon} + \Omega^q z^{p/\epsilon} x^{q/\epsilon}) = 0,$$

where Ω is any root of $x^\epsilon = 1$. The equation is therefore obtained by equating to zero the circulant determinant, whose first row has $y^{p/\epsilon}z^{q/\epsilon}$, $z^{p/\epsilon}x^{q/\epsilon}$, $x^{p/\epsilon}y^{q/\epsilon}$ as first, $(q+1)$th, $(p+1)$th elements, the other elements being zero.

By Newton's diagram, or otherwise, we see that the singularity at A, B, or C is of the same nature as the origin in the curve with Cartesian equation

$$0 = (y^q - x^p) + \sum hx^\alpha y^\beta, \tag{16}$$

h being a constant, and α, β zero or positive integers such that $\alpha q + \beta p > pq$. The expansion of y in terms of x near the origin is of the form (see H. P. A. C., Chap. VI, § 1)

$$y = x^{p/q}(a + bx^{1/q} + cx^{2/q} + \cdots). \tag{17}$$

Now suppose $d \neq 1$. First obtain the rationalized equation $f(x, y, z) = 0$ of the curve in which p and q are replaced by p' and q'. Then the required equation is

$$\prod f(\omega_1 x^{1/d}, \omega_2 y^{1/d}, z^{1/d}) = 0, \tag{18}$$

where ω_1 and ω_2 are any roots of $x^d = 1$.

It is readily seen that the singularity at A, B, C is of the same nature as the origin in

$$0 = (y^{q'} - x^{p'})^d + \sum hx^\alpha y^\beta, \tag{19}$$

while the expansion of y near the origin is still of the form given in (17). The method of page 84 of H. P. A. C. shows that the polar reciprocal of the singularity is the singularity we get, if we replace q by $p - q$.

By § 2, the point-singularities other than A, B, C are merely $\frac{1}{2}(\epsilon - 3d + 2)$ acnodes. Hence, by (15), the singularities at A, B, C are each equivalent, so far as Plücker's equations are concerned, to $\frac{1}{2}(pq - p - 3q + d + 2)$ nodes and $(q-1)$ cusps, and therefore to $\frac{1}{2}(p^2 - pq - 4p + 3q + d + 2)$ bitangents and $(p - q - 1)$ inflexions. There are besides $\frac{1}{2}(\epsilon + 4p - 2q - 3d - 6)$ other bitangents and 3 other inflexions. This statement may be verified by transforming (19) repeatedly by the substitution of xy for y or x, to analyze the singularity at the origin by the method of page 134 of H. P. A. C.

If we transform (3) by the quadratic transformation which consists in replacing x, y, z by $1/x$, $1/y$, $1/z$, we get a curve of the same type, but with $p - q$ replacing q. We may thus derive properties of one curve from those of the other, or duplicate properties of the same curve, if $p = 2q$. For instance, we see that there are exactly three conics through A, B, C which osculate the curve at a point other than A, B, C.

When the curve is projected so as to have trigonal symmetry about a point O, the circular points I, J lie on the curve unless $p + q$ is a multiple of 3. The parameters of I and J are $-\omega$ and $-\omega^2$, where $\omega^3 = 1$. If $p + q - 1$ is a multiple of 3, IJ is the tangent at I and J. If $p + q - 2$ is a multiple of 3, OI and OJ are the tangents at I and J. If $p + q$ is a multiple of 3, $t = -\omega$ and $-\omega^2$ give the point O, which is one of the acnodes. The general shape of the curve may readily be found by tracing roughly the curve with Cartesian parametral equations $x = (t-1)^p/t^q$, $y = t^{p-q}(t-1)^q$, into which it may be projected. There are four fundamentally different types, according as p and q are odd or even.

4. *Special Cases.* Some special cases may be noted.

If $p = 2$ and $q = 1$, $p = 3$ and $q = 1$, or $p = q$, the curve can be projected to have the symmetry of the equilateral triangle. Otherwise it can only have trigonal symmetry. In the first two cases there is a second triangle bearing the same relation to the curve as ABC.

If $p = 2$, $q = 1$, there is one acnode at (1, 1, 1), and three collinear inflexions given by $t = 2\cos 2\pi/9$, $2\cos 8\pi/9$, $2\cos 14\pi/9$. The points of contact of tangents from B and C, other than sides of the triangle ABC, are collinear with A, while A, B, C are "coincidence points." The points of contact of conics through A, B, C osculating the curve lie on another conic through A, B, C. Similar results hold for a non-singular cubic (H. P. A. C., p. 260, Ex. 15); but they do not hold for other values of p and q.

If $p = 3$, $q = 1$, there are three acnodes A', B', C'. Remembering the symmetry of the curve, we see that, if $A'B'C'$ is taken as triangle of reference, the equation of the curve

can be put into the form

$$y^2z^2 + z^2x^2 + x^2y^2 + 2fxyz(x + y + z) = 0.$$

To obtain the constant f, we notice that the conic through the inflexions and the conic touching the inflexional tangents (H. P. A. C., p. 275) have the property that triangles can be inscribed in the first and circumscribed to the second. This gives $f = -3/4$. The two conics are $(x^2 + y^2 + z^2) = 2(yz + zx + xy)$, $5(x^2 + y^2 + z^2) = 6(yz + zx + xy)$. The conic through the inflexions touches the sides of the triangle $A'B'C'$.

We can prove readily that, conversely, any real trinodal quartic, whose inflexions-conic touches the three lines joining the nodes, can be projected into

$$2(y^2z^2 + z^2x^2 + x^2y^2) = 3xyz(x + y + z).$$

We may also notice the case of the unicursal quintic with three ordinary cusps such that each cuspidal tangent meets the curve in another cusp, and the case of the unicursal sextic with three rhamphoid cusps. The three osculating conics through the cusps of the quintic touch at collinear points.

5. *Further Generalizations.* We may generalize our problem still further, and consider the unicursal curve in which a triangle ABC is inscribed, so that A, B, C are each given by a single value of the parameter, while the intersections of BC, CA, AB with the curve lie respectively a at C and $n - a$ at B, b at A and $n - b$ at C, c at B and $n - c$ at A.

The equations (1), (2), (3) of § 1 are replaced by

$$x : y : z = (t - 1)^a : (-t)^b(t - 1)^{n-b} : (-t)^{n-c}, \tag{20}$$

$$\epsilon \equiv n^2 - n(a + b + c) + (bc + ca + ab), \tag{21}$$

$$y^{a/\epsilon}z^{(n-a)/\epsilon} + z^{b/\epsilon}x^{(n-b)/\epsilon} + x^{c/\epsilon}y^{(n-c)/\epsilon} = 0. \tag{22}$$

It is still true that there are three inflexions other than A, B, C, and that from each of A, B, C a single tangent can be drawn other than the sides of the triangle ABC.

The parameters of the nodes other than A, B, C are given by § 2 (3), § 2 (4) being replaced by

$$\omega^\epsilon = \Omega^\epsilon = 1, \qquad \Omega^a = \omega^{n-c}, \qquad \Omega^{n-b}\omega^{b+c-n} = 1.$$

We must postpone a more detailed discussion.

BEDFORD COLLEGE, REGENTS PARK, LONDON

ON UNIFORM LIMITEDNESS OF SETS OF FUNCTIONAL OPERATIONS*

BY T. H. HILDEBRANDT

1. *Introduction.* F. Riesz † has pointed out that the method of proving the Hellinger-Toeplitz theorem is closely related to certain proofs of boundedness of sequences of integrals. Essentially these state that a sequence of limited linear operations on a given class, bounded for every member of the class, is uniformly limited. The Hellinger-Toeplitz theorem is a special case of the theorem that a double sequence of limited bilinear forms, if bounded for every member of the class, is uniformly limited. The question naturally arises whether a theorem of the second type can be derived from a theorem of the first type by an iterative process. This paper derives such a theorem of the first type, by extracting the essential feature of the linearity condition, and utilizing some of the fundamental ideas of E. H. Moore and H. L. Smith.‡

2. *The Foundations.* We shall assume a general class $\mathfrak{P}$ of elements p; also a class $\mathfrak{M}$ of functions μ on $\mathfrak{P}$ to $\mathfrak{A}$ the class of complex numbers. The class $\mathfrak{M}$ is assumed to be linear (L), i.e., for every a_1 and a_2 of $\mathfrak{A}$ and μ_1 and μ_2 of $\mathfrak{M}$, $a_1\mu_1 + a_2\mu_2$ also belongs to $\mathfrak{M}$. We assume further that it is possible to define in $\mathfrak{M}$ a distance function or norm N, which is on $\mathfrak{M}$ to the real positive part of $\mathfrak{A}$ and has the property (Δ): for every a_1 and a_2, and μ_1 and μ_2

$$N(a_1\mu_1 + a_2\mu_2) \leqq |a_1|N\mu_1 + |a_2|N\mu_2.\S$$

In terms of this norm it is possible to define the notion

* Part of a paper presented to Society, March 26, 1921.

† *Équations Linéaires à une Infinité d'Inconnues*, footnote, p. 83.

‡ *General theory of limits*, AMERICAN JOURNAL, vol. 44 (1922), pp. 102 ff.

§ This condition (Δ) is equivalent to the two conditions

$$N(\mu_1 + \mu_2) \leqq N\mu_1 + N\mu_2, \qquad Na\mu_1 = |a|N\mu_1$$

for every μ_1, μ_2 and a. For from $Na\mu \leqq |a|N\mu$ it follows for $a \neq 0$ that

$$Na\mu \leqq |a|N\mu = |a|N\left(\frac{a}{a}\mu\right) \leqq \frac{|a|}{|a|}Na\mu = Na\mu.$$

distance as $N(\mu_1 - \mu_2)$ as well as *limit*, viz.,

$$\lim_n N(\mu_n - \mu) = 0.$$

In so far as we do not postulate that $N\mu = 0$ if and only if $\mu = 0$ for every p, the limit of a sequence μ_n thus defined is not necessarily unique. If two functions μ_0 and μ_{00} are limits of the same sequence then obviously $N(\mu_0 - \mu_{00}) = 0$.

We shall call a *Cauchy sequence* a sequence $\{\mu_n\}$ such that for every $e > 0$ there exists an n_e such that if $n_1 \geqq n_e$ and $n_2 \geqq n_e$, then

$$N(\mu_{n_1} - \mu_{n_2}) \leqq e.$$

We shall assume that the class $\mathfrak{M}$ is *complete* (Cp) with respect to N, viz., that every Cauchy sequence from $\mathfrak{M}$ has a limit in $\mathfrak{M}$.*

Finally we use some of the ideas suggested by E. H. Moore and H. L. Smith. We assume another general class: $\mathfrak{Q}$ of elements q. On $\mathfrak{Q}$ we assume a binary relation $R : q_1Rq_2$, with the properties

(T) if q_1Rq_2 and q_2Rq_3 then q_1Rq_3;

(C) for every q_1 and q_2 there exists a q_3 such that q_3Rq_1 and q_3Rq_2.

Our foundation thus consists of the system:

$$\sum = (\mathfrak{A}; \mathfrak{P}; \mathfrak{M}^{\text{on } \mathfrak{P} \text{ to } \mathfrak{A}, L}; N^{\text{on } \mathfrak{M} \text{ to } \mathfrak{A} \geqq 0, \Delta}; \mathfrak{M}^{Cp}; \mathfrak{Q}; R^{\text{on } \mathfrak{Q}\mathfrak{Q}, TC}).$$

3. *The Fundamental Theorem.* Our theorem concerns a set of operations T_q on $\mathfrak{M}$ to $\mathfrak{A}$, which have the following properties:

(Δ = *triangle*), for every q, a_1, a_2, μ_1, μ_2

$$|T_q(a_1\mu_1 + a_2\mu_2)| \leqq |a_1T_q(\mu_1)| + |a_2T_q(\mu_2)|;$$

(M = *modular or limited*), there exists an M_q such that for every μ

$$|T_q(\mu)| \leqq M_qN\mu.$$

We define as the *modulus* M_q of T_q the least upper bound of the values $|T_q(\mu)|$ for $N\mu = 1$.

* The germ of such a general class $\mathfrak{M}$ is suggested by Riesz, Acta Mathematica, vol. 41 (1918), p. 72. It has been used by Bennett, Proceedings of the National Academy, vol. 2 (1916), p. 595; Lamson, American Journal, vol. 42 (1920), p. 245; Hahn, Monatshefte, vol. 32 (1922), pp. 4 ff.; Banach, Fundamenta Mathematicæ, vol. 3 (1922), pp. 135 ff.

The set T_q is *semi-uniformly bounded* over $\mathfrak{M}$,* i.e., there exists a sequence $\{q_m\}$ such that for every μ there exists an m_μ and an e_μ such that if qRq_{m_μ} then

$$|T_q(\mu)| \leqq e_\mu.$$

We note that on account of the properties T and C of R the sequence $\{q_m\}$ may always be assumed to be monotonic increasing, i.e., for every m q_mRq_{m-1}. For if the property is holding for a given sequence $\{q_m\}$ it will also hold for the sequence $\{q_m'\}$, where $q_1' = q_1$, and q_m' is such that $q_m'Rq_{m-1}'$ and $q_m'Rq_m$. Our theorem is as follows.

FUNDAMENTAL THEOREM. *A set of operations T_q on $\mathfrak{M}$ to $\mathfrak{A}$ having the properties Δ and M, which is semi-uniformly bounded over $\mathfrak{M}$, is ultimately uniformly limited, i.e., there exists a q_0 and an M such that for qRq_0*

$$M_q \leqq M.$$

This q_0 is a member of the sequence involved in the semi-uniform boundedness.

The proof of the theorem follows the lines of proofs of similar theorems in the literature.† Assume that the conclusion is not valid. Then the following statement would be true:

(*A*) For every $e > 0$ and every q_m of $\{q_m\}$ there exists a q_{me} such that $q_{me}Rq_m$ and $M_{q_{me}} > e$.

Let $\{e_n\}$ be a sequence of numbers approaching infinity. Then by a step by step process we can determine c_n, $e^{(n)}$, m_n, μ_n, q_{en}, satisfying the following conditions:

(1) $c_1 = 1$, c_n is the least of the quantities

$$\frac{e_1}{2^n M_{q_{e1}}}, \quad \cdots, \quad \frac{e_{n-1}}{2^2 M_{q_{e,\,n-1}}};$$

(2) q_{en} and μ_n such that $N\mu_n = 1$, $q_{en}Rq_{m_{n-1}}$ and

$$|T_{q_{en}}(\mu_n)| > \frac{e_n + e^{(n-1)}}{c_n};$$

(3) m_n and $e^{(n)}$ such that qRq_{m_n} implies $|T_q(\Sigma_1^n c_i\mu_i)| < e^{(n)}$.

* Cf. Moore and Smith, loc. cit., p. 114.

† Cf., for instance, Lebesgue, ANNALES DE TOULOUSE, (3), vol. 1 (1909), p. 61; Hahn, loc. cit., p. 9; Banach, loc. cit., p. 157.

The first of these is obviously possible. It insures the fact that $\Sigma_1^\infty c_n$ is convergent and

$$\Sigma_m^\infty c_n < \frac{e_{m-1}}{2M_{q_{e_{m-1}}}}.$$

For (2) we apply statement (A) to $q_{m_{n-1}}$ and $(e_n + e^{(n-1)})/c_n$. The condition (3) follows from the semi-uniform boundedness of the set T_q. From (2) and (3) and from the property Δ of T_q it follows that

$$\begin{aligned} |T_{q_{e_n}}(\Sigma_1^n c_i\mu_i)| &\geqq \Big| |T_{q_{e_n}}(c_n\mu_n)| - |T_{q_{e_n}}(\Sigma_1^{n-1} c_i\mu_i)| \Big| \\ &> c_n \frac{e_n + e^{(n-1)}}{c_n} - e^{(n-1)} \\ &= e_n. \end{aligned}$$

Now since the series $\Sigma_1^\infty c_n$ is convergent, it follows that the sequence $\Sigma_1^m c_n\mu_n$ is a Cauchy sequence. For

$$N(\Sigma_k^m c_n\mu_n) \leqq \Sigma_k^m c_n N\mu_n = \Sigma_k^m c_n \to 0.$$

Hence there exists a μ of the class $\mathfrak{M}$ such that

$$\lim_m N(\mu - \Sigma_1^m c_n\mu_n) = 0.$$

Then by the Δ property of N we can show that

$$N(\mu - \Sigma_1^m c_n\mu_n) \leqq \Sigma_{m+1}^\infty c_n.$$

Hence

$$\begin{aligned} |T_{q_{e_n}}(\mu)| &= |T_{q_{e_n}}(\mu - \Sigma_1^n c_i\mu_i + \Sigma_1^n c_i\mu_i)| \\ &\geqq \Big| |T_{q_{e_n}}(\Sigma_1^n c_i\mu_i)| - |T_{q_{e_n}}(\mu - \Sigma_1^n c_i\mu_i)| \Big| \\ &\geqq e_n - M_{q_{e_n}} N(\mu - \Sigma_1^n c_i\mu_i) \geqq e_n - M_{q_{e_n}} \Sigma_{n+1}^\infty c_i \\ &\geqq e_n - e_n/2 = e_n/2, \end{aligned}$$

i.e., since e_n approaches infinity with n, $|T_{q_{e_n}}(\mu)|$ approaches infinity with n. But from the semi-uniform boundedness of T_q, it follows that there exists an e_μ and an m_μ such that for qRq_{m_μ} it is true that

$$|T_q(\mu)| < e_\mu.$$

Since the sequence q_m is monotonic increasing and $q_{e_n} R q_{m_{n-1}}$, it would follow that the sequence $|T_{q_{e_n}}(\mu)|$ is bounded. We have thus reached a contradiction, showing that statement (A) is untenable. Hence the theorem is proved.

We can obtain various special cases of this theorem by replacing the hypotheses by special cases. If we assume, for instance, that $T_q(\mu)$ converges semi-uniformly over $\mathfrak{M}$,* then the set $T_q(\mu)$ is semi-uniformly bounded over $\mathfrak{M}$. Hence we have the following corollary.

COROLLARY I. *If T_q is a set of operations on $\mathfrak{M}$ to $\mathfrak{A}$ which have the properties Δ and M, for every q, and*

$$\lim_q T_q = T \ (\mathfrak{M};\ \textit{semi-uniformly}),$$

then there exists a q_0 and an M such that for qRq_0 it is true that

$$M_q \leqq M,$$

where M_q is the modulus of T_q. As a consequence T also has the properties Δ and M with modulus less than or equal to M.

If T is linear, i.e., if for every a_1 and a_2, and μ_1 and μ_2

$$T(a_1\mu_1 + a_2\mu_2) = a_1T(\mu_1) + a_2T(\mu_2),$$

then T has the property Δ, so that we have the following corollary.

COROLLARY II. *If T_q is a set of linear limited operations on $\mathfrak{M}$ to $\mathfrak{A}$, which is semi-uniformly convergent over $\mathfrak{M}$ to T, then the set T_q is ultimately uniformly limited and T is a limited linear operation on $\mathfrak{M}$ to $\mathfrak{A}$.*

Further specialization is possible by taking a particular class $\mathfrak{Q}$. For instance suppose $\mathfrak{Q}$ is the class of positive integers. Then the set T_q becomes a sequence of operations T_n and the semi-uniform boundedness or convergence becomes simply boundedness or convergence, respectively, for every μ. In this form the theorem includes a wide variety of cases, depending on the nature of the class $\mathfrak{M}$ and the norm N.†

We obtain another corollary of this theorem by assuming that the T_q are transformations on $\mathfrak{M}'$ to $\mathfrak{M}''$, where $\mathfrak{M}'$ and $\mathfrak{M}''$ are on $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively to $\mathfrak{A}$, and have the same properties as the class $\mathfrak{M}$, i.e., linearity, the existence of a norm with property Δ relative to which the classes are complete. It is possible to define the concepts limited or modular,

* Cf. Moore and Smith, loc. cit., p. 114.

† Hahn (loc. cit.) gives a large list of special classes and norms. The fact that a linear form, if convergent for every point of Hilbert space, is limited, is obviously also a special case of this theorem.

and semi-uniform boundedness over $\mathfrak{M}'$ for a set T_q by replacing in the definition for operations on $\mathfrak{M}$ to $\mathfrak{A}$ the absolute value signs by the norm N''. Then we have the following corollary.

COROLLARY III. *If a set of linear limited transformations T_q on $\mathfrak{M}'$ to $\mathfrak{M}''$ is semi-uniformly bounded as to $\mathfrak{M}'$, then the set is ultimately uniformly limited, i.e., there exists a q_0 and an M such that $M_q \leqq M$ or $N''T_q(\mu') \leqq MN'\mu'$ for every qRq_0 and μ'.**

4. *Bilinear Operations.* An operation B on $\mathfrak{M}'\mathfrak{M}''$ to $\mathfrak{A}$, where $\mathfrak{M}'$ and $\mathfrak{M}''$ have the properties of $\mathfrak{M}$, is *bilinear* if for every μ_1', μ_2', μ_1'', μ_2'', a_1 and a_2 it is true that

$$B(\mu_1' + \mu_2', \mu_1'' + \mu_2'') = B(\mu_1', \mu_1'') + B(\mu_2', \mu_1'') + B(\mu_1', \mu_2'') + B(\mu_2', \mu_2''),$$

and

$$B(a_1\mu_1', a_2\mu_1'') = a_1a_2B(\mu_1', \mu_1'').$$

It is *limited* or *modular*, if there exists a constant M such that for every μ' and μ''

$$|B(\mu', \mu'')| \leqq MN'\mu'N''\mu''.$$

M is called the *modulus* of B when it is the least upper bound of $|B(\mu', \mu'')|$ for all $N'\mu' = 1$ and $N''\mu'' = 1$. Then we have the following theorem.

THEOREM. *A set of bilinear limited operations B_q on $\mathfrak{M}'\mathfrak{M}''$ to $\mathfrak{A}$ which is semi-uniformly bounded over $\mathfrak{M}'\mathfrak{M}''$ is ultimately uniformly limited.*

Let $T_q(\mu')$ be the least upper bound of $|B_q(\mu', \mu'')|$ for $N''\mu'' = 1$. This will exist for every q and μ' and will have the following properties:

$$(\Delta) \quad T_q(a_1\mu_1' + a_2\mu_2') \leqq |a_1T_q(\mu_1')| + |a_2T_q(\mu_2')|,$$
$$(M) \quad T_q(\mu') \leqq M_qN'\mu',$$

* By specializing $\mathfrak{Q}$, $\mathfrak{M}'$ and $\mathfrak{M}''$, we get a set of interesting theorems, including for instance some of interest in the definition of the summability for a series or integral. Cf., for instance, Kojima, TÔHOKU JOURNAL, vol. 12 (1917), pp. 291 ff.; Carmichael, this BULLETIN, vol. 25 (1918), p. 118; Schur, JOURNAL FÜR MATHEMATIK, vol. 151 (1920), pp. 82 ff.

the set $T_q(\mu')$ is semi-uniformly bounded. For if we consider $B_q(\mu', \mu'')$ as a function of μ'' keeping μ' fixed, we can apply the fundamental theorem. If $\{q_m\}$ is the sequence involved in the semi-uniform boundedness of B_q then for every μ', there will exist an $M_{\mu'}$ and an $m_{\mu'}$ such that for $qRq_{m_{\mu'}}$ and every μ''

$$|B_q(\mu', \mu'')| \leqq M_{\mu'} N''\mu'' \quad \text{or} \quad T_q(\mu') \leqq M_{\mu'}.$$

It follows that $T_q(\mu')$ has the properties involved in the hypothesis of our fundamental theorem, and hence there exists an m_0 and an M such that for qRq_{m_0} and every μ' it is true that

$$T_q(\mu') \leqq MN'\mu'.$$

Consequently for qRq_{m_0} it is true that

$$|B_q(\mu', \mu'')| \leqq T_q(\mu')N''\mu'' \leqq MN'\mu'N''\mu''$$

which is the ultimate uniform limitedness desired.

We can make specializations, as in the case of the fundamental theorem. The Hellinger-Toeplitz theorem is a special case of this theorem if $\mathfrak{M}' = \mathfrak{M}'' =$ Hilbert space, Ω is the class of all pairs of positive integers, and

$$B_q(\mu', \mu'') = \Sigma_1^n \Sigma_1^m \beta_{ij} \mu_i' \mu_j'',$$

where β_{ij} are the coefficients of a bilinear form convergent for every μ', μ'' of Hilbert space. Obviously the sequence q_m of the semi-uniform boundedness can be taken to be the set (m, m) for all m.

Apparently, we can extend this theorem to that of a set of bilinear transformations on $\mathfrak{M}'\mathfrak{M}''$ to $\mathfrak{M}'''$, or to a set of multilinear operations or transformations. From the latter it would be possible to derive the extension of the Hellinger-Toeplitz theorem to multilinear forms.

THE UNIVERSITY OF MICHIGAN

BECK ON COORDINATE GEOMETRY

Koordinaten Geometrie. Volume 1, *Die Ebene.* By Hans Beck. Berlin, Julius Springer, 1919. x + 432 pp.

A celebrated New York divine of an earlier generation was once asked if he believed that he had a soul. He replied: "I do not understand the bearing of the question. I *am* a soul, I *have* a body."

Exactly the same distinction of verbs appears in the study of geometry. For historic or didactic reasons the student begins with synthetic geometry. The point is an undefined datum, about which more or less explicit assumptions are made. The geometry of points, lines, circles, etc., is carried to a considerable distance without any serious analytic help. After this analytic geometry is begun, and the student learns that, in order to present his points in well-bred society, he must clothe each with two numbers, called "coordinates." A moment's reflection shows him, however, that this is another case where "the apparel proclaims the man." What counts is not the undefined point, but the definite coordinates, the point is a sort of useless Kantian "thing-in-itself" that might as well be discarded.

The first writer to set this in perfectly clear light was Klein in his *Erlanger Program.* Here are various manageable groups of quantities, independent quantities, homogeneous quantities, quantities connected by certain relations. Here, further, are certain groups of transformations, which operate on these quantities; here are invariants and covariants of the groups. There are all sorts of conceivable sets of names to attach to the quantities, groups, invariants, etc. It is the path of wisdom to choose names which fit well into our geometric preconceptions and forms of speech. and to interpret in terms of them the results of our analytic manipulations, but we are at liberty at any moment to throw our vocabulary overboard and introduce new terms. Two geometries are equivalent if built on the same transformations and invariants, regardless of the shape of the letters used, or the names attached to them. At the very outset we do not say that a point *has* two coordinates, but that it *is* two coordinates. When the number field for the coordinates is real, this form of speech may seem a little artificial, but it is certainly legitimate. When, however, the number field is complex, a so-called "geometric" equivalent corresponding to the complex coordinates is bound to be more or less artificial, and the complete identification of point and coordinates by defining the former as nothing more nor less than the latter is not only natural but almost inevitable.

The present work, which takes this point of view from the outset, epitomizes and completes the results reached by a number of geometers during the last half century, the greatest of whom is Study. The fact is that the discerning reader will continually perceive the figure of Study behind a large part of what Beck has written to the extent that he may even feel inclined to exclaim:

"The voice is Jacob's voice, but the hands are the hands of Esau."

This means, of course, that the detail is very carefully done; let us look at it somewhat more closely.

In Chapter I a point is defined as a pair of complex coordinates, a line as the totality of points whose coordinates satisfy a linear equation. A sharp distinction is drawn between an isotropic and a non-isotropic line; through each point will pass just two lines of the former category. If a non-isotropic line l be given, we obtain an involutory collineation by exchanging each point of the plane with that other point whose two isotropics meet l in the same two places. This is called a *reflection* in the line l. The distance of two points is defined, analytically of course, and it is shown that distances are invariant under such reflections. Translations are obtained as the products of reflections in pairs of parallel lines, rotations as the products of reflections in intersecting lines; the whole euclidean group for the plane is built in a simple and satisfactory manner.

In Chapter II the center of interest is the straight line. The line has three homogeneous coordinates, and the idea of homogeneity is thoroughly discussed, although, curiously enough, homogeneous point-coordinates are not introduced until thirty pages later. The tangent of the angle of two lines is defined by the usual rational formula, but the tangent of the angle is not the function we want, since its period is π; we want a function with the period 2π, namely, the cosine; and this, unfortunately, is not rationally expressible in terms of the coordinates of the line. To avoid the difficulty, resort is had to the familiar process of orienting the line. Every set of numbers, not all zero, proportional to four quantities $u_0 : u_1 : u_2 : u_3$, where $-u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0$, shall be called an *oriented line* or a *spear* (German, *Speer*). This spear is said to be *on* the line $u_1x + u_2y + u_3 = 0$. The cosine of the angle of two spears is given by the formula

$$\cos(u, v) = \frac{u_1v_1 + u_2v_2}{u_0v_0}.$$

The distinction between a line and a spear is most carefully made, and it is pointed out how positive and negative distances may be rationally separated on a spear. The latter part of the chapter deals with vectors and *staffs*, i.e., vectors limited to a single line; it is a real pity that we have no good English name for these quantities except, perhaps, forces.

Chapter III is headed *Collineations*, and deals with the fundamental properties of these transformations, and with their classification. I must confess that at this point the treatment seems to me somewhat labored. It is a point of honor with the author to put every formula not only in absolutely accurate form, but in its most general form. He scorns such artifices as simplifying a problem by a change of coordinates. And when you come to think of it, he may not do this; for if you do not say that a point *has* two coordinates, but that it *is* two coordinates, why then you cannot change the coordinates and have the same point. He has such analytical skill that he suffers less from this puritanic point of view than would most writers, but he devotes fourteen very full pages to exhibiting an example of each of five general types of plane collineations, and then four more pages to the characteristic equation and the proof that all

possible types have been covered. The chapter ends with sections discussing affine transformations, stretchings, and systems of circles, oriented and non-oriented.

The fourth chapter is headed *Correlations*, but this is a misnomer. The principal use made of correlations is to define a conic according to Steiner's scheme. The bulk of the chapter is given to those collineations, happily called *automorphic*, which leave a conic in place. These are turned over on every side, and expressed in various ways, including quaternion form. What the writer is preparing for is the Cayleyan projective-metric, and this is taken up with great care. The radicals in both distance and angle formulas are avoided by the startling device of orienting both points and lines. The non-euclidean trigonometry is developed very prettily, and the already familiar euclidean metrical formulas are reached by a limiting process. A curious use of words is that whereby Lobachevski parallels, lines intersecting on the absolute conic, are defined as *paratactic;* this term was first introduced by Study in 1902 * to denote Clifford parallels, skew lines at a constant distance in three-dimensional space.

The domain in the first four chapters is the complex one. Chapter V, the last, concerns itself with real figures, real points, real collineations, real conics, etc. Circle geometry in the Gauss plane appears near the close.

And what shall we say of the book as a whole? The sketchy account above can give but the faintest idea of the amount of material; for, besides the main text, there are endless pages of problems and suggestions in small type. As mentioned above, every one of the bewildering assortment of formulas is stated with minute care, developed with conscientious rigor, and exhibited in its most general form. This is no easy accomplishment, as every geometer knows. One has the impression of a compendium, where the last word is said in deducing a large number of formulas; or, to change the metaphor, a vast arsenal where many perfect tools are built and stored for the benefit of future attacks on the difficulties of geometry. The writer could not accomplish his result without paying the price. He paid it, and the reader must pay also. The book is never obscure, the manipulations are not over-difficult, but the total effect is rather overwhelming. There are countless formulas developed for no apparent reason; there are scarcely any theorems or applications. The method is always the same: careful, straightforward conscientious algebraic analysis, with skillful use of a small number of simple algebraic identities, much as in the classical invariant theory. It is all very perfect, also very fatiguing. One has the impression that the writer is somewhat lacking in suppleness and imagination. A reviewer once said of one of Salmon's books that the reader had the instinctive feeling that the writer always hit on just the right method to solve each problem. The late Gaston Darboux once told me that it seemed to him that the essence of geometry consisted in finding in each individual problem the best method for its solution. But Beck will none of all this. Rigorous formalism must be carried through to the end, though the heavens fall.

J. L. COOLIDGE

* JAHRESBERICHT DER VEREINIGUNG, vol. 11.

KEYNES ON PROBABILITY

A Treatise on Probability. By J. M. Keynes. London, Macmillan and Co., 1921. xi + 466 pp.

Starting as a Fellowship Dissertation, interrupted by the war, preceded by economic publications widely circulated, and now published, apparently without all the finishing touches the author had liked to bestow, Keynes's *Treatise on Probability* is (see p. 433) the first systematic work in the English language on the logical foundations of probability since Venn's *Logic of Chance* (1866). Carefully documented throughout, the work closes with a 25 page bibliography which, though not pretending to completeness, is a much longer list of what has been written about probability than can be found elsewhere (p. 431). As the author has made these statements himself, the reviewer has but to quote. Two other quotations may be of interest: (1) "The theory of 'Testimony,'—of the combination of the testimony of witnesses,—has occupied so considerable a space in the traditional treatment of Probability that it will be worth while to examine it briefly. It may, however, be safely said that the principal conclusions on the subject set out by Condorcet, Laplace, Poisson, Cournot and Boole are demonstrably false. The interest of the discussion is chiefly due to the memory of these distinguished failures" (p. 180). Never perhaps since ancient biblical times has such a redoubtable army of philistines been so deftly slain (*Judges*, XV, 15). And (2) in speaking of Poincaré's opinions on probability in his *Calcul des Probabilités* and *Science and Hypothesis*, "neither of which appears to be in all respects a considered work," * we find (p. 84): "He seems to endeavor to save his reputation as a philosopher by the surrender of probability as a valid conception, without at the same time forfeiting his claim as a mathematician to work out probable formulæ of practical importance." Apparently our author little realizes that, stumbling in the dark, he has hit upon a very clever characterization of Poincaré's real philosophic system—pragmatism. The Fellowship Dissertation stage still lingers in spots.

To understand Keynes one has constantly to bear in mind that he is of the philosophic not of the mathematical school, that he derives from Leibniz, Hume and Venn rather than from Bernoulli, Laplace, and Charlier. He is seeking to lay a logical foundation for probability and to examine that foundation in the light of a wide reading of philosophic, mathematical, and statistical works on probability. He is not uninfluenced by the

* Keynes's request (p. 427) for generosity on the part of his readers is not so likely to be granted as if his own manners conformed to that spirit; besides, when writing a really critical treatise an author defeats his own ends unless he so writes that readers need not have to discount either his scientific exposition or his professional manner. If taken literally, Keynes's request must undermine our confidence in his own belief that he has accomplished the very thing that he set out to do!

Russell-Whitehead school of rigorous mathematics. Indeed, Part II, Fundamental Theorems, pp. 115–214, is much cast in their style. Much of this part is hard reading because so compact, but he would be rash indeed who would compare its difficulty with the *Principia Mathematica*; the resemblance is, fortunately for most of us, rather to the earlier *Principles of Mathematics*. Other parts are hard reading to one not used to philosophic discourse. For example (p. 36) the author is trying to be very explicit and very clear on a very important point: "A book bound in blue morocco is more like a book bound in red morocco than if it were bound in blue calf; and a book bound in red calf is more like the book in red morocco than if it were blue calf." Whether it is a deficiency in our visual or in our tactile sense or, even worse, in our ratiocinative powers which makes this statement, whether in or out of its context, unintelligible has been a distressing trilemma ever since we read it. Or is it the War or the Peace or only the Fellowship Dissertation again?

By this time not only the author but our readers will have perceived that we are not fit to review this book. There are others who appear to share our predicament. The keen mind of one of our ablest statisticians, Arne Fisher, rebels at Keynes's criticisms and shows its fire in the last section (pp. 277–279) of his *Mathematical Theory of Probabilities*, 2d edition, Macmillan, 1922. It would be interesting to quote Fisher at some length, for what an author will incorporate in a text often carries more weight than what a reviewer may write, but let the readers of the BULLETIN look up the reference and read the whole. Also let them not take too seriously what Mr. Fisher or I may say. Neither of us is primarily a philosopher, economist or logician; we have both been under the necessity, an enjoyable necessity, of using probability in a wide range of problems. The attitude toward a subject of those who use it is different from that of those who think on it. This phenomenon is well known in mathematics. The progress towards the greater use of a subject is made by those who use it, the progress towards its greater digestion and codification is made by those who think on it. Although the attitude and the method of work of these two camps are different, often so different as to be mutually incomprehensible, it is not necessary that they be mutually destructive.

Some of Keynes's comments on illustrious investigations and originators in probability might have been cast in far different language had he realized the essentially pragmatic, the unconsciously pragmatic, point of view of the scientific as contrasted with the philosophic investigator. I have recently carefully reread Laplace's *Essai philosophique sur les Probabilités*, and I cannot refrain from noting what appears to me a patient, critical, pragmatic, and tentative attitude on the part of Laplace. His treatment whether of the Law of Non-sufficient Reason, of the Law of Succession, or even of the previously mentioned "Testimony" seems open-minded and fair. But I can understand how it may seem very different to Keynes. A scientist writes a direct, clear style with definite statements; he does not qualify everything indefinitely even when he realizes that qualification would be necessary for generalization beyond the case in hand. He writes for his confrères, who understand his style. Take the case of many a

physicist who has written on that particular statistical theorem known as the Law of the Equipartition of Energy, and who has not, because he has not thought it necessary, made the statement that the law cannot possibly be true * because of the values of the ratio of the specific heats. To the philosopher or economist who may rarely make a definite statement, and who from the nature of his work and his interests perhaps cannot, the definite style of the scientist appears dogmatic, rather than pragmatic, and altogether damnable—but it is written: Judge not, that ye be not judged.

I do not, however, wish to give the impression that Keynes's book is not worth reading and study, that it was not worth doing and on the whole well done. The reason that I have dealt at such length with certain characteristics which mar the work is because, writing for those who may easily be irritated by these blemishes, I wish to explain them in the hope of reducing the irritation to such a point that the readers of this review will persevere through them to the real enjoyment and instruction that I have found in many parts of the work. The analysis of the fundamental concepts connected with the category of probability (Part I) is in many respects illuminating.† The logical treatment, in the style of Russell, of the fundamental theorems is also appealing (Part II). The general advice to "watch your step" is perhaps much less necessary to the scientist than to the philosopher because the former is dealing, as Poincaré frankly says in language only partly understood by the author, with specific cases, and perhaps only tentatively, where external indicia of impending danger or safety are not lacking, whereas the latter is developing generalities that may all too readily entice him without warning into real trouble.

There is much to interest the statistician in the discussion of the different frequency distributions resulting therefrom (pp. 194–214). Highly valuable, too, is the discussion, so far as it goes, of the normal or Gaussian law and of the Tchebycheff criterion which is so much safer than the probable error in all cases where the observations are few or are not distributed according to the normal law (pp. 351–359). The analysis of the ideas of Lexis is good but could have been carried further to include Charlier and the whole Scandinavian school who have been very active in the last forty years and who seem to be neglected by Keynes, and not by him alone. To the long bibliography one might add the essay of Royce on *The mechanical, the historical and the statistical*, SCIENCE, April 17, 1914, pp. 551–566. There are points here of philosophic import that seem not to have been covered by Keynes and might be worked into his system. Moreover some might feel that the *Statistical Mechanics* of Gibbs (1901), when viewed philosophically, particularly with respect to the meaning of equi-

* See the comment of Gibbs in the preface to his *Statistical Mechanics*, especially p. x.

† Keynes seems to belong rather to the a priori than to the a posteriori school of thought. This is natural from his philosophical, logical, analytical viewpoint. To adopt the a posteriori attitude consistently in a didactic treatise is difficult and rare; Thiele tried it relentlessly and I think that not many read his work, that fewer find it easy to read.

librium and of natural systems (whether physical or economic) and collated with the very general viewpoint of Royce and of C. S. Peirce (whose maturer work Keynes does not cite), might be worthy of at least a bibliographic reference by an author who is setting up a category of probability. However, it would be unreasonable to expect any discussion of categories to reach nearer the date of issue than about 50 years, just as one can hardly expect the full treatment of the necessary and sufficient conditions justifying a new analytical method to follow right on the heels of the introduction of such a method by the physicist (Fourier, Heaviside).

E. B. WILSON

BLASCHKE ON DIFFERENTIAL GEOMETRY

Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einstein's Relativitätstheorie. Volume 1, *Elementare Differentialgeometrie.* By W. Blaschke. Berlin, Julius Springer, 1921. x + 230 pp.

This volume is the first of a series of three which the author plans to publish under the first part of the title given above. It is devoted in part to the classical theories of the differential geometry of curves and surfaces, and in part to some very interesting special chapters of these theories with which the author himself has been especially occupied. In his preface Professor Blaschke says that this first volume contains a presentation of the properties of curves and surfaces which are invariant under the group of motions, that the second will be devoted to affine differential geometry, and that a third will present the geometrical theories of Riemann and Weyl, which are so closely related to the Einstein theory of gravitation.

There are two interesting features of the book that attract one's attention at the very start. The first is the consistent use of vector notations. All of us who have lectured on differential geometry have doubtless been impressed with the great economies in presentation which these notations would afford. I have myself hesitated to use them in lecture courses because of the loss of time necessitated at the beginning of a course by explanations to hearers who have had no experience with vector notations, and because of the slight element of awe and mystification which these notations seem to arouse in the minds of those who have had only a limited acquaintance with them. After reading Professor Blaschke's book, I have grave doubts of the correctness of my attitude. The properties of vectors which one needs in differential geometry are few and simple, and he has demonstrated that they may be clearly and concisely explained as the occasions for their use arise. When one considers the many applications of vector analysis in other domains as well as in geometry, it seems clear that we should acquaint our students with the elements of the subject at the earliest possible moment. Mathematical physicists usually shy away from abstract mathematical notations, as they did from the non-euclidean theories of space before the recent revolution. Is it not curious that they should be the leading advocates of the vector analysis notations

which mathematicians in equally appropriate situations have in many cases studiously avoided?

The second outstanding feature of the book is the fifth chapter, devoted to the macroscopic properties of surfaces as contrasted to the differential properties with which most of us are more familiar. The only simply closed analytic surfaces with constant Gaussian curvature are spheres. If we designate an analytic simply closed convex surface by the name oval surface, then the only oval surfaces with constant mean curvatures are spheres. On every oval surface there are at least three closed geodesics and the image of each of them on the Gaussian sphere of the surface bisects the area of the sphere. These are samples of the theorems which this chapter contains. At the present time comparatively few results in the macroscopic theory of surfaces have been attained, most of them being properties of oval surfaces. The proofs which lead to them, as Professor Blaschke says in his preface, are among the most difficult known in any branch of analysis. It seems to me that the author has presented these proofs on the whole with great clearness, and we may feel confident that in this domain, in which he himself has had great interest, a comprehensive summary of known results has been given.

Throughout the book one notices frequent applications of the theory of the calculus of variations. This is inevitable in one form or another in the treatment of geodesics and minimal surfaces, but the author is well acquainted with the theory and apparently welcomes opportunities to apply this knowledge. In no case are theorems used, however, which are not established in the text. One may look forward to still further interesting applications of the calculus of variations in the third volume promised by the author.

The book is written in concise and "snappy" style, but the sequences of logical steps are clear and the text always interesting. References to original sources and historical remarks are frequent. The latter are sometimes more than mere paragraphs. On pages 80–81 and 123, for example, are very interesting historical sketches devoted to Monge and Gauss. In a number of places in the book computations are left to the reader. These are sometimes of considerable extent, but as I read the book I found only a few instances in which additional suggestions from the author would have helped to a more rapid understanding. At the ends of chapters are lists of problems and of theorems to be proved by the reader. These are in many cases not elementary, but references are usually given. I should regard the lists as on the average unsuited to the needs of beginning students, but suggestive of much interesting study and reading outside the limits of the book itself for readers of all types.

Chapter I is devoted to vector notations and the differential geometry of twisted curves, with applications to Bertrand curves, "Böschungslinien," evolutes and involutes, and isotropic curves. Böschungslinien are curves whose tangents make a constant angle with a fixed direction in space, and the author determines all such on a sphere or a paraboloid of revolution. In § 9 a foretaste of the later macroscopic chapter is given by a proof of the theorem that every oval plane curve has at least four vertices, that is, four

points where the rate of change of the curvature with respect to the length of arc is zero. One misses, in this chapter, the theory of the moving trihedron whose axes lie in the tangent, principal normal, and binormal of a given twisted curve, and by means of which the properties of many curves associated point for point with the given one can be very simply deduced.

Chapter II is in many respects a calculus of variations chapter. The author begins by showing how the binormal and first curvature κ of a twisted curve can be defined by the first variation of the length integral along a curve, and considers the more general problem suggested by Radon of minimizing an integral of the form $I = \int \varphi(\kappa) ds$ where s is the length of arc. He then studies the isoperimetric properties of the circle, proving that the circle is actually the curve of given length enclosing a maximum area by the methods of Frobenius and Hurwitz. One wonders a little at this point why this material is introduced here. But it turns out that in Chapter VI the author gives two very interesting generalizations, for minimal surfaces and for the sphere, of the formula which is fundamental in Frobenius' proof, and the more general results of the very elegant proof of Hurwitz are also applied in a later section. The last three sections of the chapter are devoted to two theorems of Schwarz concerning minimum properties of twisted curves with constant curvature $\kappa = 1$.

In the seventy pages of Chapters III and IV the author presents more of the theory of surfaces than I should have thought possible in such a limited space. In the former of them one finds the theories of lines of curvature, asymptotic lines and conjugate systems, the Gaussian sphere, the formulas of Gauss and Codazzi, and related subjects. The latter is devoted to the elements of the theory of applicable surfaces, geodesics and related coordinate systems, Beltrami's differential invariants, isothermal parameters and conformal representations of one surface on another. Although the presentation is concise it seems to me unusually readable and interesting.

I have mentioned already the character of the contents of Chapter V, which is one of the most interesting in the book. There are in all some nine macroscopic theorems, six concerning oval surfaces, two concerning closed surfaces, and one concerning surfaces of constant curvature $K = -1$. In § 86 the author describes in some detail a problem which is still unsolved. On every geodesic line there corresponds in general to every point A a conjugate point A'. Every arc AB not containing A' is shorter than all other arcs near it joining A and B, while arcs AB containing A' between A and B do not have this property. On a sphere the geodesics are great circles and the conjugate point A' is diametrically opposite to A on the surface. The problem of § 86 is to determine the surfaces on which the lengths of the geodesic arcs AA' are all the same. It seems likely that these surfaces are necessarily spheres, but the author at present knows no complete proof of this surmise. He describes in very interesting fashion the results attained and the difficulties remaining to be overcome.

Chapter VI is entitled "Extreme bei Flächen" and contains an introduction to the theory of minimal surfaces and a study of the isoperimetric properties of the sphere. For minimal surfaces the equations of Weierstrass

and Study are derived, besides Schwartz's formula expressing the area of a minimal surface as a line integral along its contour, and the necessary condition for a minimum which he deduced from the second variation of the area integral on the surface and which corresponds to Jacobi's condition in the calculus of variations. A discussion of the problem of the determination of a minimal surface by a curve and a strip of normals on the surface, with some allied theorems, is also given. Among all surfaces with given area the sphere is the one which encloses a maximum volume, but it is not a simple matter to prove this property. The author uses a process of symmetrization of oval surfaces due to Steiner in order to prove that the maximizing surface if it exists must be a sphere. But there still remains the necessity of proving that a minimizing surface actually exists. Existence proofs have been made by Schwartz (1884), Minkowski (1903), Blaschke (1916), and Gross (1917). The proof here given is the last one, which is a modification of that of Blaschke. In the course of this proof the relation $O^3 - 36\pi V^2 \geqq 0$ is deduced for the surface area O and the volume V of an oval surface, the equality holding only when the surface is a sphere. This is a generalization of a similar relation $L^2 - 4\pi F \geqq 0$ used in an earlier chapter in the Frobenius proof of the isoperimetric property of a circle, L being the length of a plane oval and F the area which it encloses. As a second generalization the author shows also that Frobenius' formula holds for a simply closed curve in space provided that L is the length of the curve and that F is interpreted as the surface area of the minimal surface passing through the curve.

The final chapter of the book is devoted to line geometry, especially the theory of ruled surfaces and congruences. Every straight line is uniquely determined by a unit vector $\boldsymbol{a}$ and a second vector $\boldsymbol{b}$ orthogonal to the first, by means of the equations $\boldsymbol{x} \times \boldsymbol{a} = \bar{\boldsymbol{a}}$, where $\boldsymbol{x} = (x_1, x_2, x_3)$ is a variable point on the line. By the equation $\mathfrak{A} = \boldsymbol{a} + \epsilon\bar{\boldsymbol{a}}$ the straight line furthermore determines a unique vector $\mathfrak{A}$ with components in the linear algebra of Clifford whose elements have the form $\boldsymbol{a} + \epsilon\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ being real numbers and ϵ a unit such that $\epsilon^2 = 0$. It is readily provable that every such vector $\mathfrak{A}$ is a unit vector, and conversely every unit vector $\mathfrak{A}^2 = (\boldsymbol{a} + \epsilon\boldsymbol{b})^2 = 1$ has its real vectors $\boldsymbol{a}$, $\bar{\boldsymbol{a}}$ satisfying $\boldsymbol{a}^2 = 1$, $\boldsymbol{a}\bar{\boldsymbol{a}} = 0$, and therefore determines uniquely a straight line in space. By this most interesting device of Study the lines of space are set into one-to-one correspondence with the points on a unit sphere in the Clifford space, the Clifford space being the totality of points $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3)$ whose coordinates $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ are numbers in the Clifford algebra. A rotation in the Clifford space determines uniquely a motion in the original $\boldsymbol{x}$-space, and conversely, so that the determination of the invariants under rotations in the former is equivalent to the determination of invariants under the group of motions in the latter. A ruled surface is now defined by a vector $\mathfrak{A}(t)$ whose elements are functions of a single parameter t, and a congruence by a vector $\mathfrak{A}(u, v)$ depending upon two parameters. It is altogether fascinating to see how the usual properties of these surfaces and congruences fall out naturally and readily under this new aspect.

G. A. Bliss

SHORTER NOTICES

Introduction to the Mathematical Theory of the Conduction of Heat in Solids. By H. S. Carslaw. Second edition, completely revised. London, Macmillan and Co., 1921. xii + 268 pp.

The first edition of Professor Carslaw's work appeared in 1906 under the title *Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat.* This book, which was reviewed in volume 15 of this BULLETIN, was divided into two parts corresponding to the two principal topics in the above title. It had been out of print for some time, and the author wisely decided to rewrite the entire book in such fashion as to take account of the progress that has been made in the theories involved since the appearance of the first edition. The very considerable expansion of size inevitably involved in such a rewriting has led naturally to the replacing of the two parts of the first edition by the two volumes of the second edition.

The first volume, which was entitled *Introduction to the Theory of Fourier's Series and Integrals*, was reviewed in volume 28 of this BULLETIN. As pointed out in that review, it showed extensive rewriting and considerable expansion as compared with the first part of the original edition. The present volume likewise exhibits considerable alteration from the second part of the first edition. Chapters I–VI, which contain substantially the same material as the corresponding chapters of the first edition, have been extensively revised; chapters VII–X contain much new material; chapters XI and XII are entirely new.

The first edition of the book was particularly noteworthy from the fact that it was the first work in English in which a large group of the standard problems that arise in the theory of the conduction of heat were dealt with systematically in a rigorous fashion. In addition to this group, however, certain other problems were introduced where the possibility of a rigorous treatment was merely indicated instead of being carried out. This was probably due partly to considerations of space, and partly to a desire to keep the book sufficiently elementary to be palatable to the student of applied mathematics, the omitted proofs being in many cases long and difficult. In some cases the omission was perhaps due to the fact that no completely rigorous discussion was available in the literature. In the present edition such of the missing proofs as are not unusually lengthy are supplied, as for example in sections 17 and 31; in most other instances, where the complete discussion is not given, suitable references are furnished which will enable the reader, if so disposed, to find a rigorous discussion in the literature. One rather noteworthy exception to this rule may be found in section 51, where the possibility of the expansion of an arbitrary function of three variables in a triple Fourier's series is assumed. The omission of any reference in this case is quite understandable, as in spite of the very extensive literature on the ordinary Fourier's series, the reviewer is unaware of any discussion of the triple Fourier's series in existing literature which would supply the missing proof in the section mentioned.

Such a discussion will be shortly available, however, in a paper * by Miss Bess Eversull, which is to appear in the ANNALS OF MATHEMATICS.

Quite a little of the new material to be found in the present edition of Professor Carslaw's book is based on his own contributions to the subject since the appearance of the first edition. Thus for example the discussion in section 90 of the flow of heat in a wedge is based on a paper by the author in the PROCEEDINGS OF THE LONDON SOCIETY ((2), vol. 8 (1909–10)). Also chapter XI, dealing with the use of contour integrals in the solution of the equation of conduction, is based on papers by the author in the PHILOSOPHICAL MAGAZINE ((6), vol. 39 (1920)) and the PROCEEDINGS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY (vol. 20 (1921)).

Chapter XII, which deals with the use of integral equations in the solution of the equation of conduction, is only a very brief sketch of this subject. It was doubtless introduced mainly for the purpose of acquainting the student of applied mathematics with the possibilities in this direction, and in this manner stimulating him to a study of the works that deal with the subject in a more extensive manner. A list of such works is given in a footnote at the beginning of the chapter.

The bibliography of works on the conduction of heat, found in Appendix II, has been brought up to date by the addition of titles of articles that have appeared since the publication of the first edition. The added titles for this period of fifteen years occupy three pages and form approximately one-fourth of the entire list, which corresponds to a period of a century. Thus we have a rough index of the mathematical activity in this particular field during recent years.

CHARLES N. MOORE

Praxis der Gleichungen. By C. Runge. Zweite, verbesserte Auflage. Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1921. 2 + 172 pp.

This book, printed in 1900 as number XIV of the SAMMLUNG SCHUBERT, now appears in larger format and better type as one of the new GÖSCHENS LEHRBÜCHEREI,—the second of the first group (*Reine Mathematik*). Changes have been few and unimportant. The reviewer noticed a paragraph at the foot of page 98 and a figure between pages 106 and 107, neither of which appears in the first edition, and an index has been added. Two errata have been carried over from the first edition: in line 2 from the bottom of page 38, for a_{14}, read a_{14}/a_{11}; in line 3 from the bottom of page 87, for 3217.18, read 5217.18. A new misprint is the omission of the i in the exponent of e in line 11 of page 165. But, considering the nature of the contents, there are on the whole remarkably few typographical errors.

This volume has become such a classic that it is scarcely necessary to describe it more fully than to say it gives in considerable detail, illustrated by numerous examples, the actual processes to be followed in solving numerical equations where the roots are to be found to a high degree of accuracy. Some theory is given, culminating in Sturm's Theorem, but the emphasis is laid on actual computations. It is to be hoped that

* This paper was presented to this Society at its meeting in Chicago, April 14, 1922. See this BULLETIN, vol. 28 (1922), p. 289.

sometime an English translation will make this work even more accessible to our students. A study of it is most heartily recommended to the mathematician who is interested in the applications of theory to numerical practice, to the applied mathematician who has to perform computations, and to the teacher of secondary mathematics who thinks that with Horner's method the last word on the solution of equations has been said.

ELIJAH SWIFT

Les Applications Élémentaires des Fonctions Hyperboliques à la Science de l'Ingénieur Électricien. By A. E. Kennelly. Paris, Gauthier-Villars, 1922. vii + 151 pp.

This book covers the subject of a series of lectures delivered in France by the author as exchange professor, representing a group of American universities. The same topic has been covered by Professor Kennelly in a previous work (in English) published in 1912 by the University of London Press. The differences are so slight that an American student would naturally prefer the one written in his own language.

The aim of both books is to spread more widely the fact that cable problems are best understood in terms of hyperbolic functions,—of real variables for the case of direct current with distributed leakage, of complex variables in the case of alternating currents with distributed earth or parallel-wire capacitance. This aim is very heartily to be commended. The author's method of solving alternating current problems by considering the corresponding problem with direct current, and then by replacing resistances by complex impedances is also easily the best. In some text-books this method is not used as consistently and thoroughly as it should be. It enables the principles of the wave-filter, for example, to be stated very easily by means of the two principles of the addition of impedances (admittances) of series (parallel) conductors. It is probable, however, that this method finds its way more frequently into lectures than into text-books. In this connection the reviewer would advise a more frequent use of four-terminal impedances (current between AB in terms of voltage drop from C to D).

Either of the two books would also be a useful source of problems in the elementary theory of the complex variable.

In some ways the book under review suffers from condensation. In particular, the remark that the hyperbolic angle of a sector can be measured, not only by twice its area, but also "par la longueur de l'arc hyperbolique AP, rapportée à la longueur croissante du vecteur OP" is not clear until one observes, in the earlier book, that by this is meant the integral with respect to the arc of the reciprocal of the radius.

The book under review contains a brief study of transient currents which is not contained in the earlier work.

Finally mathematicians may have their attention called to Appendix F of the earlier book in which is shown an interesting relation between continued fractions of a certain class and hyperbolic functions. The student is referred to a paper by the author in the ANNALS OF MATHEMATICS, vol. 9 (1908), p. 85 et seq.

P. J. DANIELL

Cours de Physique Mathématique de la Faculté des Sciences. Compléments au Tome III. *Conciliation du Véritable Déterminisme Mécanique avec l'Existence de la Vie et de la Liberté Morale.* Par J. Boussinesq. Paris, Gauthier-Villars, 1922. xlviii + 217 pp.

Strange title for the third volume of a book on Mathematical Physics! Strange assertion in a strange title! Yet appearing in all simplicity in the closing volume of the new edition of an older publication, now the mature judgment of a distinguished scientist over eighty years of age, the affirmation in fact of a first statement made forty-five years ago! Happy the person who can more confidently state at eighty years of age what he thought was true when he was thirty-five! Member of the Institut, author of many treatises, many memoirs, and minor papers, professor of mathematical physics and of the theory of probabilities—who has a better right to state what he finds, with confidence and serenity?

What is the startling assertion? It is very simple and yet it makes much of scientific philosophy rock preparatory to its fall in ruins. This assertion is that mechanics declares the necessity of a directive principle which is outside mechanics in order to account for the phenomena of the universe. What! Can it be that the most definite of all the branches of physics, itself the most definite and well-understood of the sciences, the branch which even gives its name to that mode of thinking which passes among some scientists as the only straight and narrow path to truth—the mechanistic philosophy—can it be that this discipline itself has turned traitor to the cause, and gone over to the camp of the enemy where freedom smiles, where there are directive principles, entelechies, even perhaps real mind? Is this the outcome of a lifetime study of the laws of mechanics and probabilities? Yet so it is. And indeed he is not without very respectable company in these opening years of a new century, now almost a quarter gone.

The investigation of M. Boussinesq has all the clarity of French mathematics, is easy to read, is not—as he says distinctly—a metaphysical argument, but is the unescapable conclusion of a mathematical consideration of the laws of motion. It is simply this in brief: the laws of motion are stated always in the form of differential equations. These have in many cases, when we come to integrate them, ambiguities in the integrals, due to the fact that the characteristic curves of integration are bifurcated at certain points and also to the fact that these curves often have envelop curves that represent the singular solutions. A particle traveling on a characteristic curve—to take the simplest case—may follow the envelop, for each characteristic curve somewhere is tangent to the enveloping curve. What determines then the path chosen in these cases, since the laws of motion show that the paths are indeterminate, and yet the particles do follow unique paths? The only answer is that at the singular points where a choice of path is possible there must be a directive principle which selects the path. This does not affect the accelerations, therefore does not consist of a force, and hence is not to be accounted for in the energy equations. This becomes the true distinctive character of phenomena due to life, conformably to the extreme physico-chemical instability of living beings,

not imitated by inanimate things. The configuration of the particles of the universe is not altogether explainable by the previous configurations. There were in some situations alternative possibilities, and the laws of nature do not explain why one choice was made rather than the other,—in fact if this could be explained, there would have been no choice and no singularity in the characteristic curves.

The idea that all the phenomena of physics and chemistry can be reduced to modes of motion vanished long ago, for it became evident that the "field" played a very important part in the explanation of phenomena studied by the physicist and chemist, and the field again is described by means of partial differential equations. More complicated singular solutions have to be admitted, as well as singularities of the surfaces of the integrals. This would add force to what M. Boussinesq has to say regarding total differential equations as representing phenomena. It is evident in either case that there may be situations in which the laws of physics—all of which can be stated in mathematical form—fail to account for the following events. This simple fact is enough to invalidate mechanistic hypotheses. For these hypotheses have for their specific aim to do without any other laws than those of physics.

Boussinesq calls attention to a memoir of Poisson's which contains the incipient notion of a directive principle, arising from a consideration of singular solutions. In his *Traité de Mécanique* however he does not mention the matter. Again Duhamel reproduces Poisson's example but does not make much of it. Cournot also considered the matter, but evidently was prejudiced in favor of determinism and sought to find justification for throwing out the singular solutions and the bifurcated paths. Everything considered, there seems to be demanded a dynamics of the directive agencies, which will lead directly from the mechanics of forces to the activities of life, mind, and society. Among the principles for such a science are the principle of simplicity, and that of unity, the principle of variation, and of continuity, the principle of economy or least action. Among the problems he hopes this new science may some day solve is that of the marvelous adaptations of organ to function, of species to environment, usually dismissed with the mere statement that they are necessary for existence.

The volume has several long complementary notes, related to the main memoir. Such topics are the analogy of the mechanism of life with wave-motion, the dissipation of energy and the reversal of purely mechanical movements, the rôle and legitimacy of geometric intuition, the application of the "threshold" of sensation to a possible theory of certain quanta, and a consideration of the impossibility of knowledge of intramolecular phenomena.

This volume is most interesting for those who have philosophic instincts, not only for its positive contribution, but for the fact that it opens the way to a clearer understanding of the freedom that is found to lurk even in mathematical equations. In these days when the pendulum is swinging towards a better interpretation of the facts of life and mind, it is well to comprehend just wherein real determination finds its sphere.

JAMES BYRNIE SHAW

NOTES

The following university courses in mathematics are announced for the academic year 1923–1924:

UNIVERSITY OF CHICAGO.—Courses which continue for more than one quarter are indicated with Roman numerals, as I, II, III, IV. By Professor E. H. Moore: Hermitian matrices in General Analysis, I, II, III, IV; Vectors, matrices, and quaternions.—By Professor L. E. Dickson: Hypercomplex numbers, I, II; Theory of equations.—By Professor H. E. Slaught: Differential equations; Definite integrals; Elliptic integrals; Advanced calculus.—By Professor G. A. Bliss: Theory of functions of a real variable; Calculus of variations; Integral equations; Advanced calculus.—By Professor E. J. Wilczynski: Seminar on Geometry; Metric differential geometry; Solid analytic geometry.—By Professor F. R. Moulton: Modern theories of analytic differential equations, I, II; Advanced ballistics, I, II.—By Professor W. D. MacMillan: Analytic mechanics, I, II; Celestial mechanics; Dynamics of rigid bodies; Theory of the potential.—By Professor A. C. Lunn: Vector analysis; Applications of vector analysis in the theory of electromagnetism; Thermodynamics; Vector analysis in Riemann-Einstein space.—By Professor J. W. A. Young: Selected topics in mathematics.—By Dr. Mayme I. Logsdon: Theory of functions of a complex variable; Introduction to higher algebra. Courses in research are also offered by Professor Moore in Foundations of mathematics and in General Analysis, by Professor Bliss in Analysis, by Professor Dickson in Algebra and Theory of numbers, by Professor Wilczynski in Geometry, and by Professor Lunn in Applied mathematics.

COLUMBIA UNIVERSITY.—By Professor T. S. Fiske: Differential equations.—By Professor F. N. Cole: Theory of groups (first term).—By Professor D. E. Smith: History of mathematics; Practicum in the history of mathematics.—By Professor C. J. Keyser: Modern theories in geometry (first term); Introduction to mathematical philosophy (first term).—By Professor Edward Kasner: Einstein's theory of gravitation.—By Professor W. B. Fite: Infinite series (second term).—By Professor J. F. Ritt: Elliptic functions (first term); Analytic theory of numbers (second term).—By Dr. G. A. Pfeiffer: Topics in projective geometry (second term).—By Dr. Jesse Douglas: Topics in higher geometry (second term).

CORNELL UNIVERSITY.—By Professor J. H. Tanner: Mathematics of finance.—By Professor Virgil Snyder: Algebraic geometry.—By Professor F. R. Sharpe: Hydrodynamics and Elasticity.—By Professor Arthur Ranum: Line geometry.—By Professor W. B. Carver: Advanced calculus.—By Professor D. C. Gillespie: Theory of functions of a complex variable.—By Professor W. A. Hurwitz: Differential equations of mathematical physics.—By Professor C. F. Craig: Projective geometry.—By Professor F. W. Owens: Advanced analytic geometry.—By Professor H. M. Morse: Analysis situs (first term); The restricted problem of three bodies (second term); Elementary differential equations.—By Dr. G. M. Robison: Calculus of variations (first term); Infinite series (second term).—By Mr. D. S. Morse: Modern higher algebra.

HARVARD UNIVERSITY.—By Professor W. F. Osgood: Advanced calculus; Theory of functions (second course).—By Professor J. L. Coolidge: Line geometry (first half-year); Probability (second half-year); Subject matter of elementary mathematics (first half year); Kinematics (second half-year).—By Professor G. D. Birkhoff: Space, time, and relativity (first half-year); Advanced dynamics and quantum theory (second half-year).—By Professor E. V. Huntington: The fundamental concepts of mathematics (first half-year).—By Professor O. D. Kellogg: Dynamics (second course); Theory of potential functions (first half-year); Theory of point sets (second half-year).—By Professor W. C. Graustein: Modern geometry; Projective geometry (first half-year); Geometrical transformations (second half-year).—By Dr. J. L. Walsh: The partial differential equations of mathematical physics (second half-year).—By Dr. Philip Franklin: The analytical theory of heat and problems in elastic vibrations (second half-year); Relativity, advanced course (second half-year). There will also be a seminar in analysis conducted by Dr. Walsh and Dr. Franklin, and the following courses of research: Topics in the theory of functions, Professor Osgood; Topics in postulate-theory, Professor Huntington; Topics in geometry, Professor Coolidge; Topics in the theory of potential functions, Professor Kellogg; Topics in the theory of differential equations, Professor Birkhoff; Topics in geometry, Professor Graustein.

UNIVERSITY OF ILLINOIS.—By Professor E. J. Townsend: Functions of a complex variable; Differential equations and advanced calculus.—By Professor G. A. Miller: Theory of groups; Theory of equations and determinants (first semester).—By Professor J. B. Shaw: Vector methods.—By Professor A. B. Coble: Projective geometry.—By Professor R. D. Carmichael: Linear difference equations.—By Professor A. Emch: Algebraic surfaces; Constructive and projective geometry (second semester).—By Professor A. R. Crathorne: Statistics (first semester); Actuarial theory.—By Professor G. E. Wahlin: Calculus of variations (second semester).—By Professor A. J. Kempner: Theory of numbers.—By Professor H. Blumberg: Graphical and numerical methods (second semester); Introduction to modern mathematics.—By Professor E. B. Lytle: Teacher's course (first semester); Fundamental concepts of mathematics (second semester).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.—By Professor F. S. Woods: Advanced calculus and differential equations; Higher geometry.—By Professor C. L. E. Moore: Theoretical aeronautics; Rigid dynamics.—By Professor H. B. Phillips: Thermodynamics; Statistical mechanics; Quantum theory; Theory of the gyroscope.—By Professor Joseph Lipka: Analytical mechanics; Mathematical laboratory.—By Professor F. L. Hitchcock: Application of mathematics to Chemistry.—By Dr. George Rutledge: Modern algebra; Theory of functions.—By Dr. J. S. Taylor: Mathematics of investment.—By Dr. N. Wiener: Fourier's series and integral equations.—By Dr. S. D. Zeldin: Vector analysis.

UNIVERSITY OF PENNSYLVANIA:—By Professor E. S. Crawley: Higher plane curves.—By Professor G. H. Hallett: Infinite series and products (first term); The theory of functions of a complex variable (second term).

—By Professor O. E. Glenn: The theory of invariants.—By Professor F. H. Safford: The mathematical theory of elasticity.—By Professor G. G. Chambers: Synthetic projective geometry.—By Professor H. H. Mitchell: The analytic theory of numbers.—By Professor M. J. Babb: Introduction to the theory of numbers.—By Professor F. W. Beal: Linear differential equations of the second order (first term); Advanced calculus (second term).—By Professor J. R. Kline: Pointset theory (first term); Integral equations (second term).

PRINCETON UNIVERSITY:—By Professor J. H. M. Wedderburn: Complex variables.—By Professor L. P. Eisenhart: Differential geometry.—By Dr. C. E. Hille: Advanced course in analysis.—By Dr. C. C. MacDuffee: Algebraic invariants.—By Dr. B. Kerekjarto: Analysis situs.—By Professor O. Veblen: Seminar on mathematical physics.

At the request of the Committee on Endowment of this Society, Professor J. W. Young has consented to take charge of the general publicity for that committee.

During the first semester of the academic year 1923–24, Professor B. A. Bernstein of the University of California will be on leave of absence, and will reside in New Haven, Conn. During his absence from California, Professor D. N. Lehmer will act in his stead as secretary of the San Francisco Section of the Society.

The Polish Academy of Sciences and Letters of Cracow celebrated its fiftieth anniversary on June 16, 1923.

By a special arrangement recently made, members of the Deutsche Mathematiker-Vereinigung who contribute two dollars ($2.00) yearly will receive the JAHRESBERICHT, complete in both parts. Membership is open to mathematical departments as units, as well as to individuals. The usual membership with the regular dues, which are at present 600 marks per year, carries with it merely the subscription to the second part of the JAHRESBERICHT, containing business notes, reports of meetings, reviews, problems and solutions, etc.; it is to be hoped that those joining on this basis will voluntarily add a small subscription to cover postage. Applications for membership should be sent to Professor L. Bieberbach, Berlin-Schmargendorf, Marienbaderstrasse 9.

Mr. L. J. Mordell, of Cambridge University, has been appointed a Fielden professor of pure mathematics at the University of Manchester. Professor Mordell (cf. this BULLETIN, April, 1923, p. 189) will give two courses this summer at the University of Chicago.

Dr. W. A. Hamilton will be on the staff of the University of Wisconsin for the coming year as Lecturer in Mathematics. For the past ten years he has been head of the department of Mathematics in Beloit College, and for the past five years Registrar.

Assistant Professor E. P. Lane, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Chicago.

Assistant Professor W. L. Crum, of Yale University, has been appointed assistant professor of statistics in the department of economics at Harvard University.

Professor Charles Niven, of the University of Aberdeen, died May 11, 1923, at the age of seventy-eight years.

NEW PUBLICATIONS

I. PURE MATHEMATICS

ARCHIMEDES. Kugel und Zylinder. (Ostwald's Klassiker der Exakten Wissenschaften, Nr. 202.) Leipzig, Akademische Verlagsgesellschaft, 1922. 80 pp.

BALLY (E.). Principes et premiers développements de géométrie synthétique générale moderne. Paris, Gauthier-Villars, 1922. 8 + 218 pp.

BROWNE (R. T.). The mystery of space: a study of the hyperspace movement in the light of the evolution of new psychic faculties, and an inquiry into the genesis and essential nature of space. London, Routledge, 1921. 8vo.

DE COMBEROUSSE (C.). See ROUCHÉ (E.).

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